Spectral Shift Function for the magnetic Schrödinger operators

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Definition of SSF

For two self-adjoint operators H_0 and H on a separable Hilbert space $\mathcal{H}$, M. G. Kreĭn's spectral shift function (SSF) $\xi(\lambda) = \xi(\lambda; H, H_0)$ for the pair (H, H_0) is defined by the trace formula

$$\mathrm{tr}\,[f(H) - f(H_0)] = \int_{\mathbb{R}} \xi(\lambda) f'(\lambda) d\lambda \left(= -\int_{\mathbb{R}} \xi'(\lambda) f(\lambda) d\lambda \right)$$

for any $f \in C_0^\infty(\mathbb{R})$ (cf. Kreĭn 1953). If the spectrum of H and H_0 are included in $(-c, \infty)$ and $(H + cI)^{-m} - (H_0 + cI)^{-m}$ is in the trace class for some $c \in \mathbb{R}$, then SSF is defined via the equality

$$\xi(\lambda; H, H_0) = \begin{cases} -\xi((\lambda + c)^{-m}; (H + cI)^{-m}, (H_0 + cI)^{-m}) & (\lambda > -c), \\ 0 & (\lambda \leq -c). \end{cases}$$

SSF for Schrödinger operators

In the case $H_0 = -\Delta$ and $H = -\Delta + V$ on $\mathbb{R}^d$, a well-known sufficient condition for the existence of SSF is

$$|V(x)| \le C\langle x\rangle^{-\rho}, \qquad \langle x\rangle = (1+|x|^2)^{1/2} \tag{1}$$

for some $C > 0$ and $\rho > d$. It is also known that regularized SSF can be defined under more mild decaying condition depending on the dimension d. At least we always need the short range condition $\rho > 1$.

Nice reviews: Birman-Yafaev 1992, Birman-Pushnitski 1998, Yafaev 2007.

Birman-Kreĭn formula

When $\rho > d$, the scattering matrix $S(\lambda) = S(\lambda; H, H_0)$ exists and $S(\lambda) - I$ is in the trace class. Then the Birman-Kreĭn formula

$$\det S(\lambda) = \exp(-2\pi i \xi(\lambda)) \tag{2}$$

holds for almost every $\lambda > 0$ (Birman-Kreĭn 1962). If we write the eigenvalues of $S(\lambda)$ as $e^{2i\delta_{\lambda,n}}$ $(n = 1, 2, \ldots)$, we have

$$\xi(\lambda) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \delta_{\lambda,n}. \tag{3}$$

The RHS of (3) is absolutely summable when $\rho > d$. The number $\delta_{\lambda,n}$ is called the phase shift when V is radial, since $\delta_{\lambda,n}$ is just the asymptotic phase shift of some generalized eigenfunction for H.

Magnetic Schrödinger operator

Next we consider the magnetic Schrödinger operator on $\mathbb{R}^2$

$$H = \left(\frac{1}{i}\nabla - A\right)^2, \quad A = (A_1, A_2).$$

The corresponding magnetic field and the total magnetic flux are

$$B = \operatorname{curl} A = \partial_1 A_2 - \partial_2 A_1, \quad \alpha = \frac{1}{2\pi}\int_{\mathbb{R}^2} B(x)dx.$$

If the vector potential A satisfies

$$|A(x)| + |\operatorname{div} A(x)| \le C\langle x\rangle^{-\rho}, \quad \rho > 2, \tag{4}$$

then we can also define SSF $\xi(\lambda; H, H_0)$ ($H_0 = -\Delta$) in a similar way. However, (4) never holds when $\alpha \neq 0$, and we cannot define SSF in the ordinary manner.

Main result

Nevertheless, we can define similar quantity even if $\alpha \neq 0$, in the following sense.

Theorem 1

Assume the magnetic field B is a real-valued C^1 function on $\mathbb{R}^2$ such that

$$|B(x)| \leq C\langle x\rangle^{-\rho}, \quad \rho > 3.$$

Let $\alpha = \int_{\mathbb{R}^2} B(x)dx/(2\pi)$ be the total magnetic flux, and H_α be the Schrödinger operator for the Aharonov-Bohm magnetic field

$$H_\alpha = \left(\frac{1}{i}\nabla - A_\alpha\right)^2, \quad A_\alpha = \alpha\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

with the regular boundary condition at $x = 0$.

Main result

Theorem 1 (continued)

Then, there exists a vector potential A with $\operatorname{curl} A = B$, such that

$$\begin{aligned} &\lim_{R\to\infty} \operatorname{tr}\left[\chi_R(f(H) - f(H_0))\chi_R\right] \\ = \quad & -\frac{1}{2}\{\alpha\}(1-\{\alpha\})f(0) + \int_{\mathbb{R}} \xi(\lambda; H, H_\alpha) f'(\lambda) d\lambda \end{aligned}$$

for every $f \in C_0^\infty(\mathbb{R})$. Here χ_R is the characteristic function of the disc $\{|x| \le R\}$, $\{\alpha\} = \alpha - [\alpha]$ is the fractional part of α, and $\xi(\lambda; H, H_\alpha)$ is the ordinary SSF for the pair (H, H_α).

Similar results:

- Borg 2006 (Ph. D. thesis) $f = e^{-t\lambda}$, $H = H_\alpha$, with Dirichlet b.c.
- Tamura 2008 $f' = 0$ near the origin, χ_R is replaced by the smooth cut-off function.

SSF for Aharonov-Bohm magnetic field

The above result is formally interpreted as

$$\xi(\lambda; H, H_0) = \xi(\lambda; H, H_\alpha) + \xi(\lambda; H_\alpha, H_0),$$

$$\xi(\lambda; H_\alpha, H_0) = \begin{cases} \frac{1}{2}\{\alpha\}(1 - \{\alpha\}) & (\lambda > 0), \\ 0 & (\lambda \leq 0). \end{cases}$$

The eigenvalues of the scattering matrix $S(\lambda) = S(\lambda; H_\alpha, H_0)$ are $e^{i\alpha\pi}$ and $e^{-i\alpha\pi}$ (∞-deg.) (Ruijsenaars 1983, Adami-Teta 1998, Roux-Yafaev 2002). Then Birman-Kreĭn formula becomes

$$\begin{aligned} \xi(\lambda; H_\alpha, H_0) \quad = \quad & \left(-\frac{\{\alpha\}}{2} - \frac{\{\alpha\}}{2} - \frac{\{\alpha\}}{2} - \cdots \right) \\ & + \left(\frac{\{\alpha\}}{2} + \frac{\{\alpha\}}{2} + \frac{\{\alpha\}}{2} + \frac{\{\alpha\}}{2} + \cdots \right). \end{aligned}$$

This equality does not make sense at all, but it also suggests us there is some cancellation mechanism.

Outline of Proof

The key tool for the proof of Theorem 1 is Pochhammer's generalized hypergeometric function

$$ {}_pF_q\left(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + n) \cdots \Gamma(\alpha_p + n)}{\Gamma(\beta_1 + n) \cdots \Gamma(\beta_q + n)} \frac{z^n}{n!}. $$

Here we obey E. M. Wright's notation. The asymptotic formula for ${}_pF_q$ has been studied from the beginning of 20th century (cf. Barnes 1907, Wright 1935, 1940, Braaksma 1962, Luke 1969, 1975, ...). The asymptotic formula consists of algebraic series and exponential series, whose coefficients can be explicitly calculated (at least by Mathematica).

Outline of Proof

Proposition 2

Let $0 < \alpha < 1$ and $f \in C_0^\infty(\mathbb{R})$. Then, we have

$$\mathrm{tr}\left[\chi_R\left(f(H_\alpha) - f(H_0)\right)\chi_R\right] = \int_0^\infty \xi_{\alpha,R}(\lambda) f'(\lambda) d\lambda,$$

$$\xi_{\alpha,R}(\lambda) = -F_\alpha(\sqrt{\lambda}R) - F_{1-\alpha}(\sqrt{\lambda}R) + F_0(\sqrt{\lambda}R) + F_1(\sqrt{\lambda}R),$$

$$\begin{aligned} F_\nu(z) &= \frac{z^{2\nu+4}}{8\sqrt{\pi}} {}_2F_3\left(\nu+1, \nu+\frac{3}{2}; 2\nu+2, \nu+3, \nu+3; -z^2\right) \\ &\quad + \frac{z^{2\nu+2}}{4\sqrt{\pi}} {}_2F_3\left(\nu+\frac{1}{2}, \nu+1; 2\nu+1, \nu+2, \nu+2; -z^2\right). \end{aligned}$$

Outline of Proof

Combining Proposition 2 and the asymptotic formula for ${}_2F_3$, we obtain more detailed asymptotics of the function $\xi_{\alpha,R}$ as follows.

Proposition 3

Let $0 < \alpha < 1$. Then we have

$$\begin{aligned}\xi_{\alpha,R}(\lambda) &= \frac{1}{2}\alpha(1-\alpha) - \frac{\sin\alpha\pi}{4\pi}\frac{\cos(2\sqrt{\lambda}R)}{\sqrt{\lambda}R} \\ &\quad + \frac{(2\alpha+1)(2\alpha-3)\sin\alpha\pi}{16\pi}\frac{\sin(2\sqrt{\lambda}R)}{(\sqrt{\lambda}R)^2} + O((\sqrt{\lambda}R)^{-3}),\end{aligned}$$

as $\sqrt{\lambda}R \to \infty$.

The principal term coincides with Tamura's one, but the next term differs because of the difference of the formulation (Tamura uses the smooth cut-off).