Continuous Time Quantum Walk on finite dimensions

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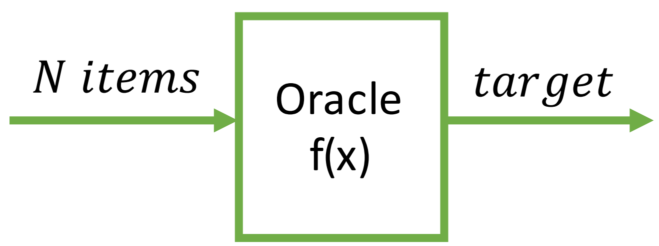
Grover Algorithm: Unstructured Search



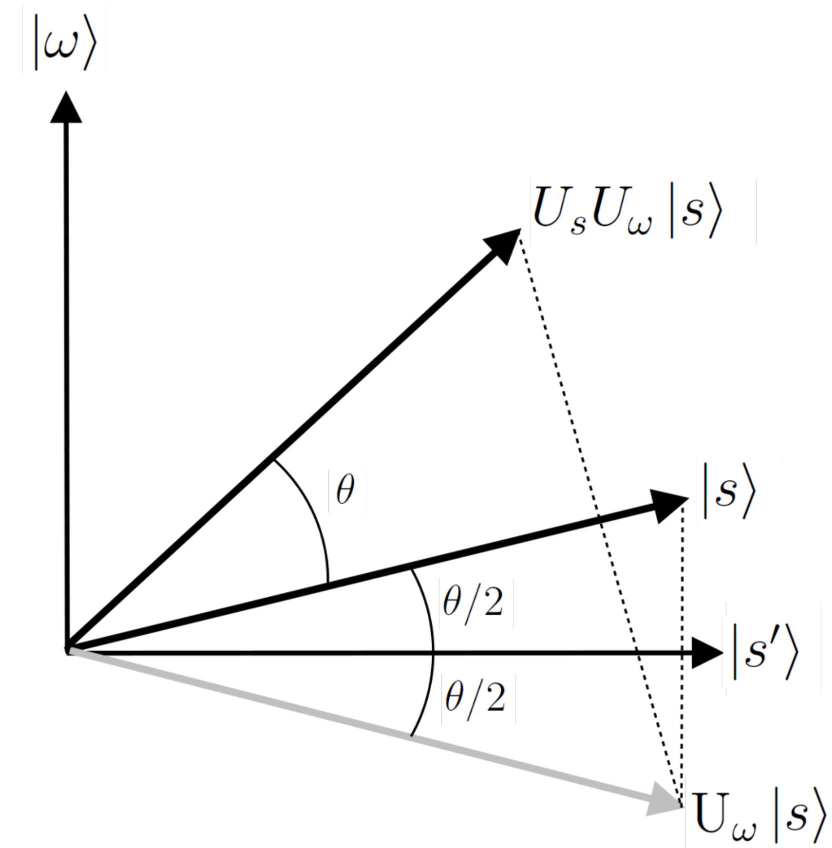
$$f(x) = \begin{cases} 1 & x = w \\ 0 & otherwise \end{cases}$$

Initialize the system to the state

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

Apply Grover Iteration

$$(U_s U_w)^{\sqrt{N}} |s\rangle$$



$$U_s = 2|s\rangle\langle s| - I$$

$$U_w = I - 2|w\rangle\langle w|$$

$$\theta = 2 \arcsin \frac{1}{\sqrt{N}}$$

Quantum Walk Basics for Spatial Search

Random Walk

$$\frac{d}{dt}p_x = \sum_{y=0}^{N-1} L_{xy}p_y$$

Continuous time quantum walk

$$\frac{d\Psi_x(t)}{dt} = \sum_y H_{xy}\Psi_y(t)$$

$$H = \gamma L - |\omega\rangle\langle\omega|$$

initial state $|s\rangle = \frac{1}{N}\sum_x |x\rangle$

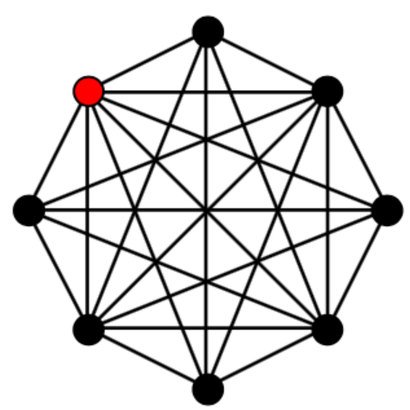
marked state $|w\rangle$

Unstructured search:

f(x) is a computable function

Spatial Search:

N items stored in a d-dimensional physical space

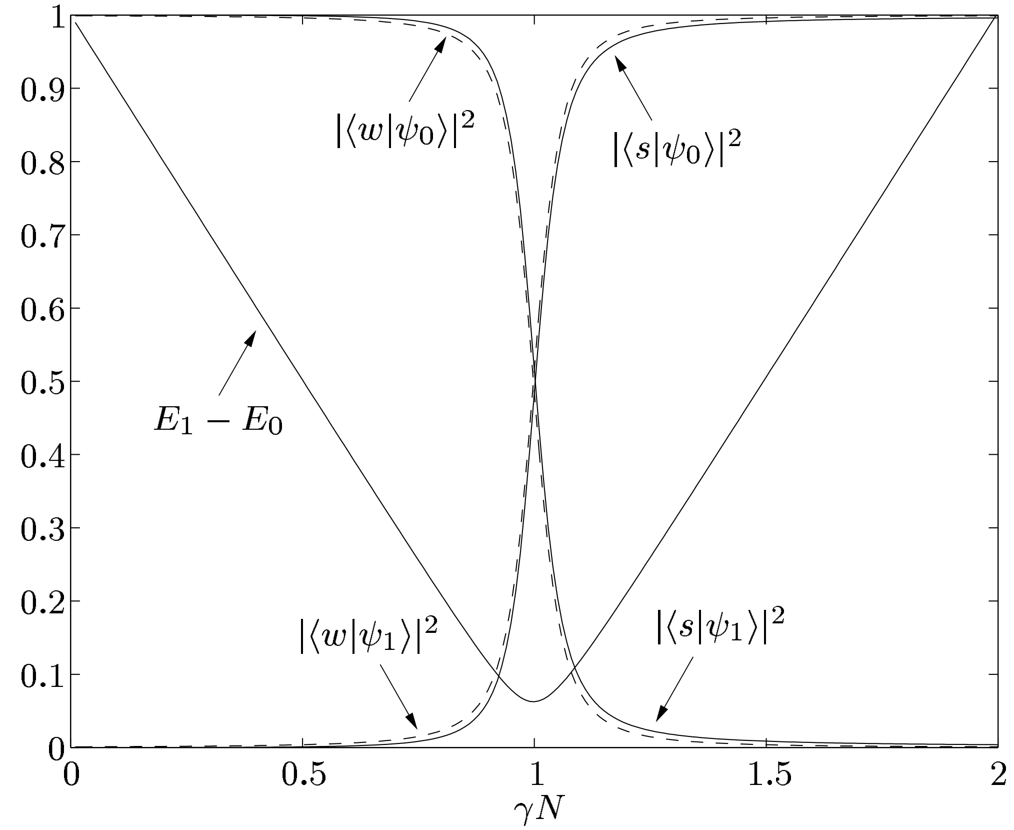


Laplacian = Degree Matrix - Adjacency Matrix

$$L = D - A$$

Critical Point in the Hamiltonian

$$\gamma = 0$$
$$H = -\left|w\right\rangle\left\langle w\right|$$
$$0^{th}, 1^{st} = \left|w\right\rangle, \left|s\right\rangle$$



$$\gamma = \infty$$
$$H = \gamma L$$
$$0^{th}, 1^{st} = \left|s\right\rangle, \left|w\right\rangle$$

$$\gamma = \gamma_c$$
$$0^{th}, 1^{st} = \left(\left|w\right\rangle \pm \left|s\right\rangle\right)/\sqrt{2}$$
$$T = \frac{\pi}{2}\sqrt{N}$$

A. Childs et al. Spatial Search by Quantum Walk (2004)

CTQW: optimal performance

Quadratic Speedup $O\left(\sqrt{N}\right)$ **Grover efficiency**

- complete graph, hypercube, strongly regular graph
  (E. Farhi and S. Gutmann 1998, A. M. Childs et al 2002, J. Janmark et al, 2014)

- Erdös Renyi graph $p \geq log^{\frac{3}{2}} N/N$
  (SS. Chakraborty et al, 2016)

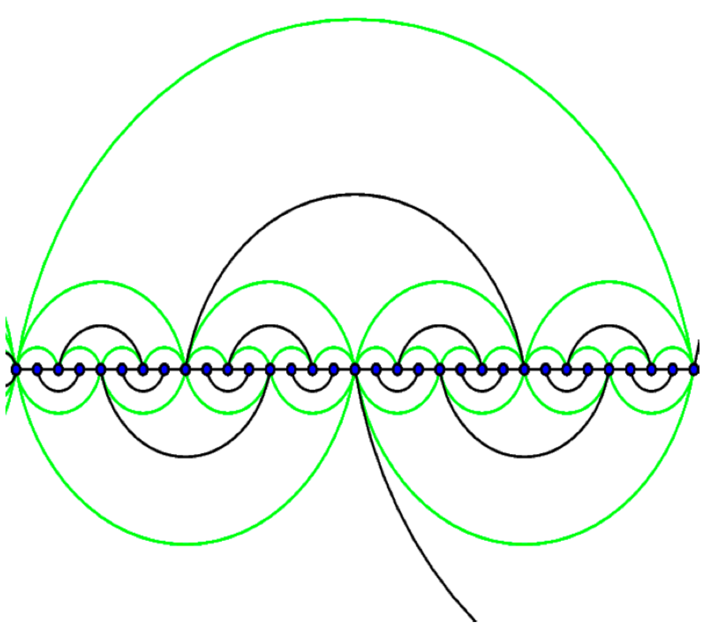
- lattices $d > 4$
  (A. M. Childs et al 2004)

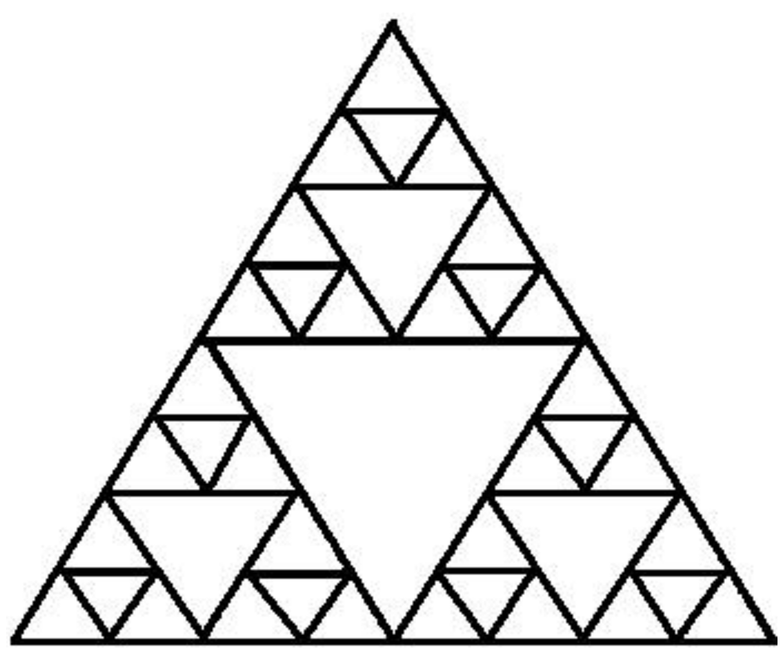
- We extend CTQW to fractal graphs with real fractal dimension
- Spectral dimension of graph Laplacian determines the computational complexity

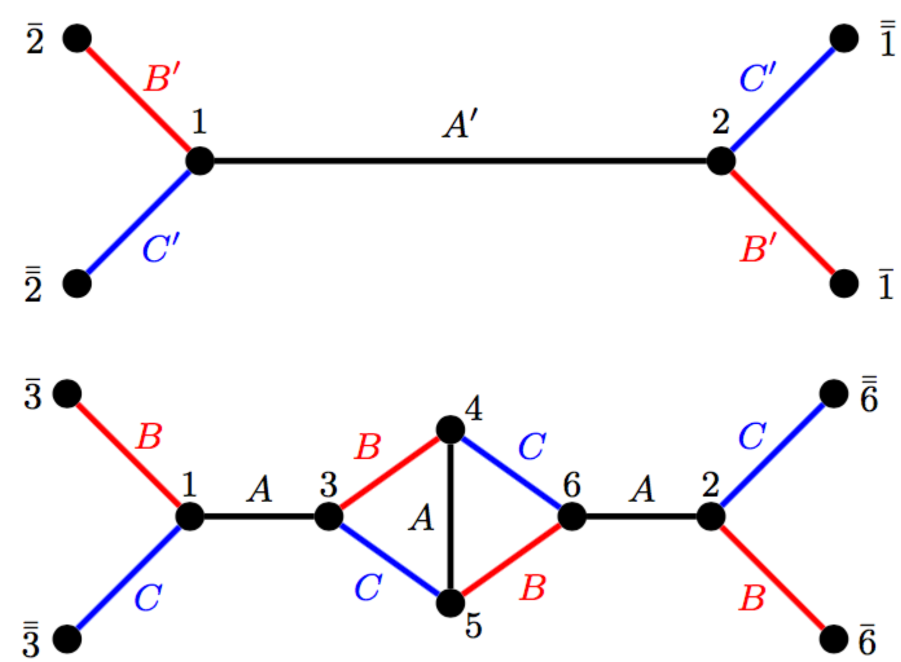
Finite Dimensional Fractals

we generalize to arbitrary real (fractal) dimension

- Hierarchical networks
- Sierpinski Gasket, Migdal-Kadanoff network with regular degree 3
- Diamond fractals based on the Migdal-Kadanoff renormalization group scheme







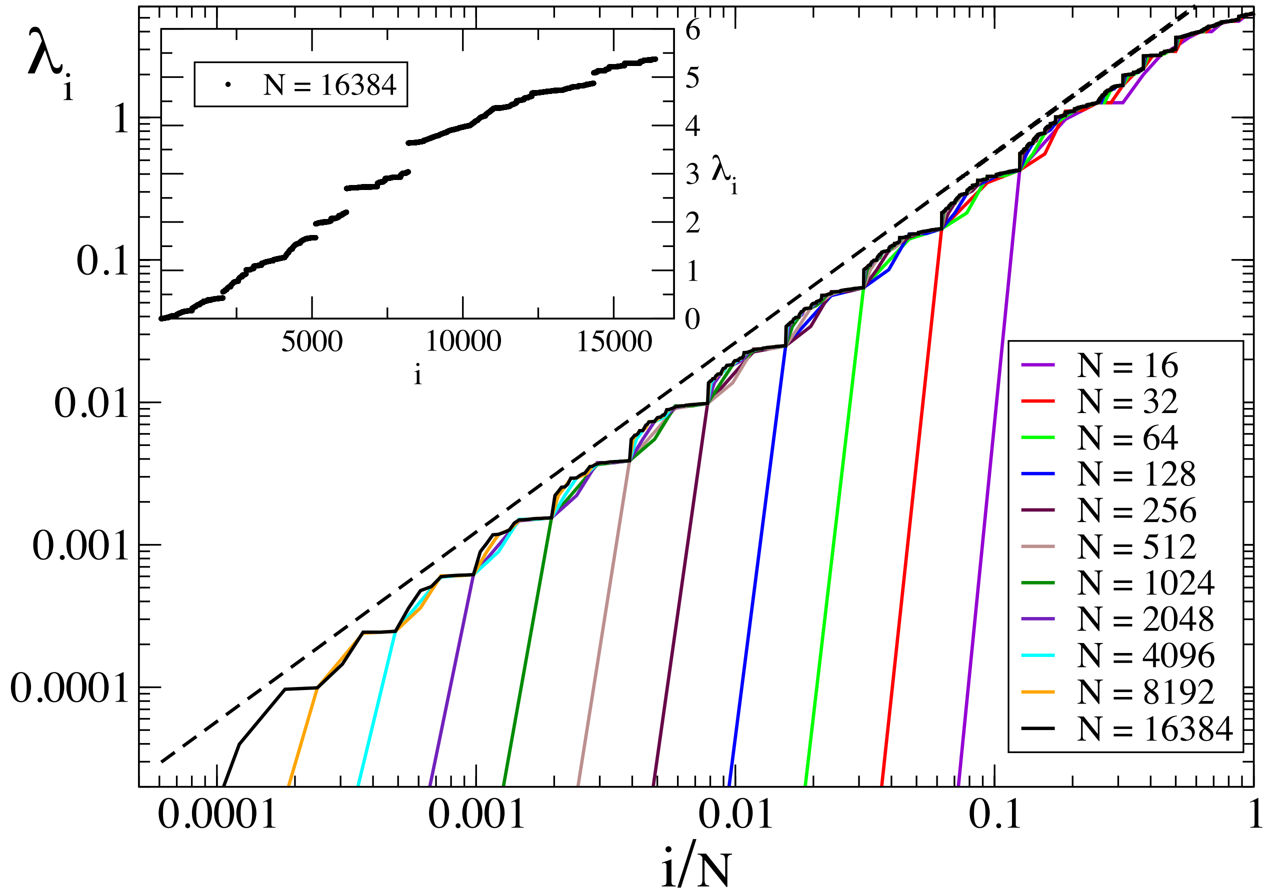
Dimensions in fractal networks

Fractal dimensions

$$N \sim l^{d_f}$$

Spectral dimension

$$\lambda_i \sim N^{-2/d_s}$$

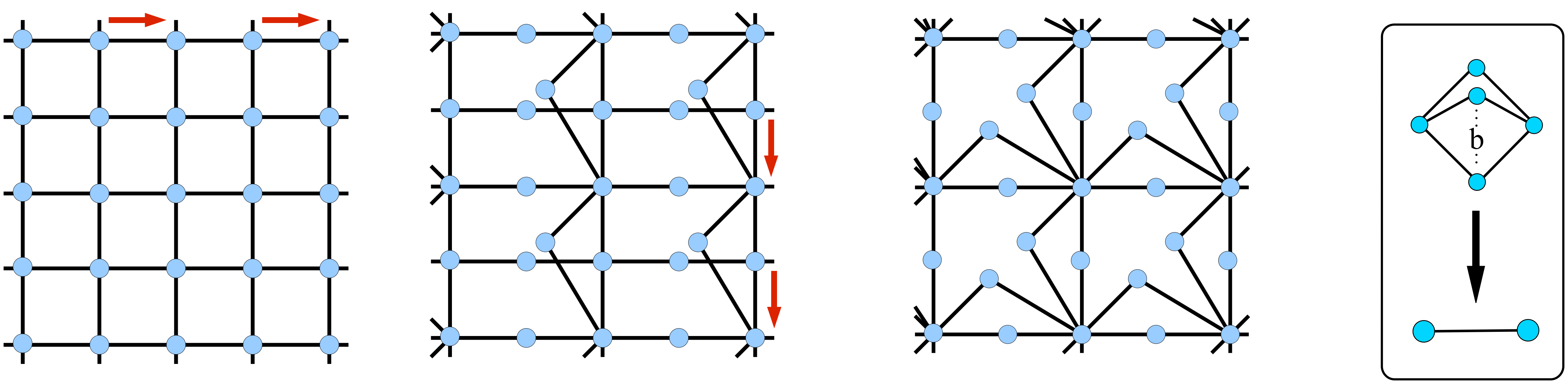


Hierarchical Network with regular degree 3

Migdal-Kadanoff renormalization group(MKRG)

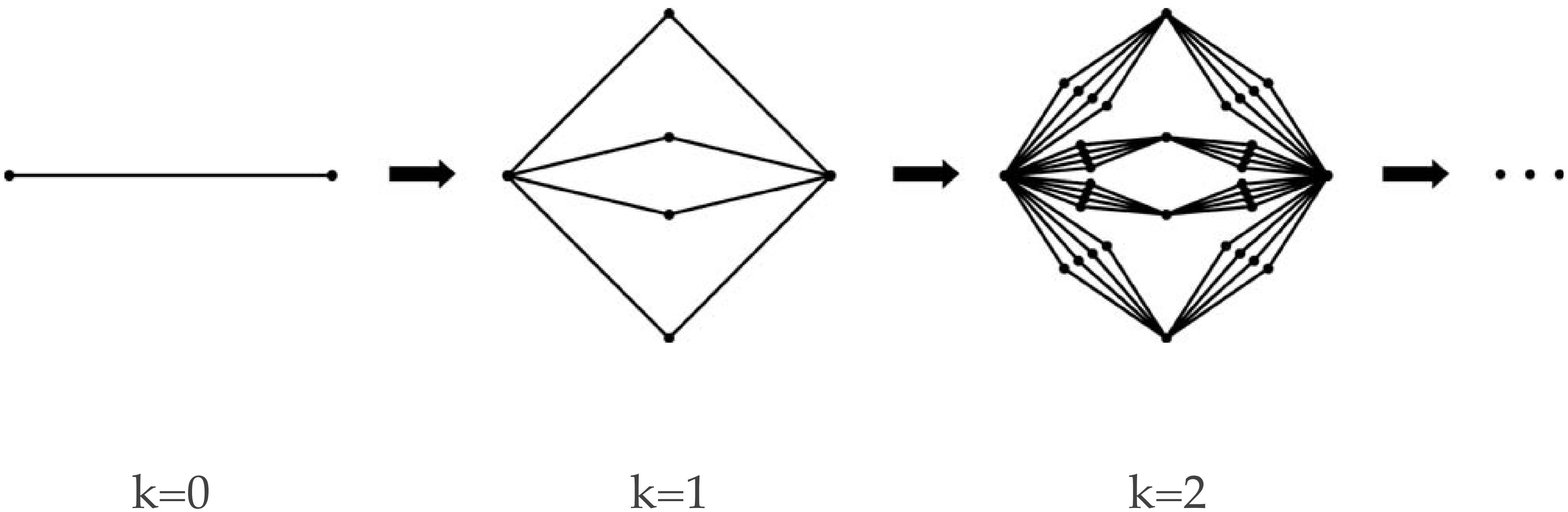
Model regular lattices closely $d_s = d_f = d$

arbitrary real dimension



Bond-moving scheme on square lattices with rescaling length l=2,
branching factor b=2

Procedure to build the Diamond Fractals



$$d_f = d_s = d = 1 + \frac{\ln b}{\ln 2}$$

Measure Critical Point

The spectral Zeta function

$$I_j = \frac{1}{N} \sum_{i=1}^{N-1} (\frac{1}{\lambda_i})^j$$

When the CTQW is optimal for search, the critical point takes place (numerically true for almost all sites in fractals we consider)

$$\gamma_c = I_1$$

The transition probability

$$\Pi_{s,\omega} = \frac{I_1^2}{4I_2} \sin^2(\frac{2I_1}{\sqrt{I_2}} \frac{t}{\sqrt{N}})$$

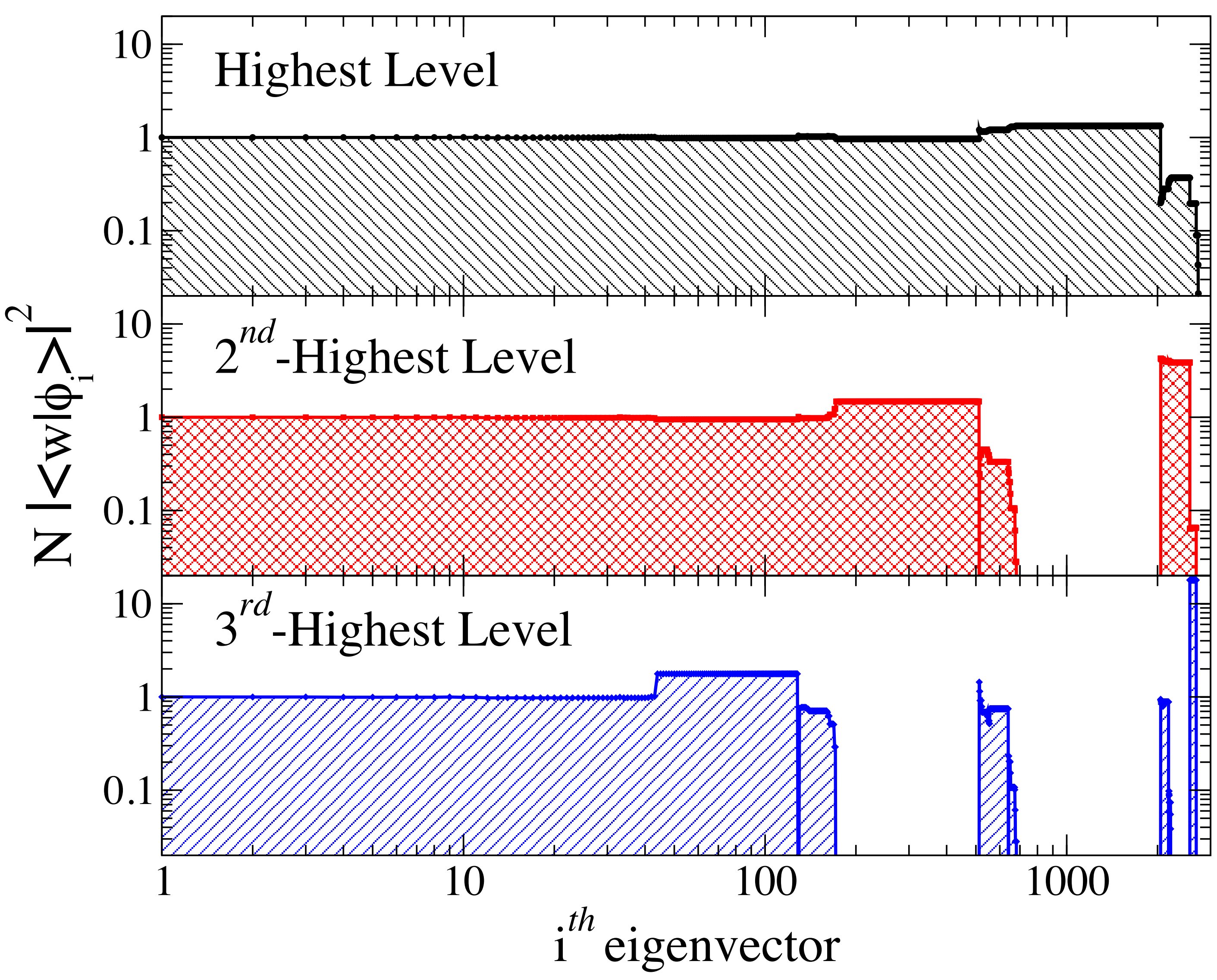
A. Childs et al. Spatial Search by Quantum Walk (2004)

Assumption on fractal Laplacian eigenvector

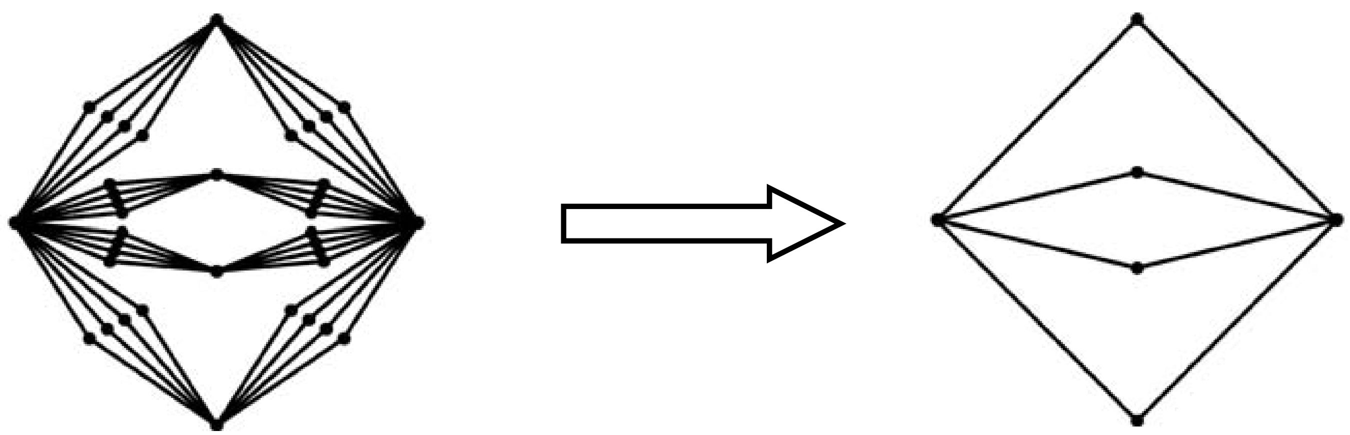
$$I_j = \sum_{i=1}^{N-1} \frac{|\langle \omega | \phi_i \rangle|^2}{\lambda_i^j}$$

$$|\langle \omega | \phi_i \rangle|^2 \sim \frac{1}{N}$$

MK renormalization group with b=2



Renormalization Group Argument



$$det\left[L^{(k)}\left(q_i^{(k)}, p_i^{(k)}, \cdots\right)\right] \qquad det\left[L^{(k+1)}\left(q_i^{(k+1)}, p_i^{(k+1)}, \cdots\right)\right]$$

The spectral Zeta function

$$I_j \sim \left(\frac{\partial}{\partial \epsilon}\right)^j \ln\left[\frac{1}{\epsilon} det\left(L+\epsilon\right)\right] |_{\epsilon\to 0}$$

$$\sim \begin{cases} N^{\frac{2j}{d_s}-1} & d_s < 2j \\ const & d_s > 2j \end{cases}$$

Computational Complexity of CTQW

$$\gamma_c \sim \begin{cases} N^{\frac{2}{d_s}-1}, & d_s < 2 \\ const, & d_s > 2 \end{cases} \qquad T \sim \begin{cases} N^{\frac{1}{2}}, & d_s > 4 \\ N^{\frac{1}{2}} \ln^{\frac{3}{2}} N, & d_s > 2 \\ \gtrsim N^{\frac{2}{d_s}}, & 2 < d_s < 4 \end{cases}$$

- spectral dimension of network Laplacian determines the computational complexity
- complement the discussions on regular lattices and mean-field networks

References:

- Shanshan Li and Stefan Boettcher, arXiv preprint arXiv: 1607.05317 , 2016
- Stefan Boettcher and Shanshan Li, arXiv preprint arXiv: 1607.05168 , 2016

Thank you !

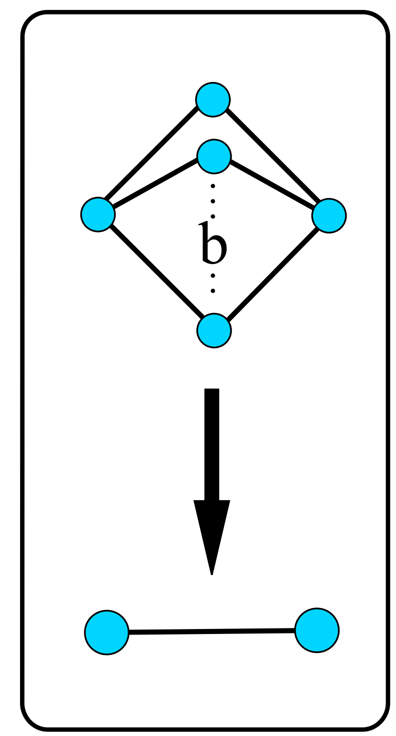
RG calculation for spectral determinant

$$\frac{1}{\sqrt{\det L}} = \int \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{\pi}} \exp\{-\sum_{n=1}^{N}\sum_{m=1}^{N} x_n L_{n,m} x_m\}$$

$$B_i(x,y) = C_i \exp\{-\frac{q_i}{2}x^2 - \frac{q_0}{2}y^2 + 2pxy\} \Rightarrow$$

$$B'_{i-1}(x,z) = \int \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{b} \frac{dy_i}{\sqrt{\pi}} B_i(x,y_j) B_1(y_j,z) \Rightarrow$$

$$B'_{i-1}(x,z) = C'_{i-1} \exp\{-\frac{q'_{i-1}}{2}x^2 - \frac{q'_0}{2}y^2 + 2p'xz\}$$



$$\det(L+\epsilon)|_{\epsilon\to 0} = C_k^{-2} b^{2k} (q_k^2/4 - p_k^2)$$

• Stefan Boettcher and Shanshan Li, arXiv preprint arXiv: 1607.05168 , 2016