Non-existence of the wave operators for the repulsive Hamiltonians

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We consider the Schrödinger equation with the pair of Hamiltonians given by

$$H_0 = p^2 - x^2, \quad H = H_0 + V \quad \text{on } L^2(\mathbb{R}^n) \tag{1}$$

where $p = -i\nabla$ is the momentum and x^2 means $|x|^2$. The interaction potential $V \in L^\infty(\mathbb{R}^n)$ is real-valued multiplication operator. V will vanish when $|x|$ is so large.

$$V(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{2}$$

Known result

2005 Bony-Carles-Häfner-Michel[1]

Under the following decay condition:

$$|V(x)| \leqslant C(\log\langle x\rangle)^{-1-\epsilon}, \quad \epsilon > 0 \tag{3}$$

where $\langle x\rangle = (1 + x^2)^{1/2}$*, they proved the existence of the wave operators and their asymptotic completeness.*

[1] *J. Math. Pures Appl.* **84** (2005)

Cook-Kuroda method

$$\int_1^\infty \|Ve^{-itH_0}\phi\|_{L^2}dt < \infty \tag{4}$$
$$\Longrightarrow \operatorname*{s-lim}_{t\to\infty} e^{itH}e^{-itH_0} \quad \textit{exists.}$$

$$\begin{aligned}
&\|(e^{it_1H}e^{-it_1H_0} - e^{it_2H}e^{-it_2H_0})\phi\|_{L^2} \\
&= \|\int_{t_2}^{t_1} \partial_t(e^{itH}e^{-itH_0})\phi dt\|_{L^2} \\
&\leqslant \int_{t_2}^{t_1} \|Ve^{-itH_0}\phi\|_{L^2}dt \longrightarrow 0 \quad \text{as} \quad t_1, t_2 \longrightarrow \infty
\end{aligned} \tag{5}$$

The case of the free Schrödinger operator : $p^2 = -\Delta$

If we write the decay condition on V as

$$|V(x)| \leqslant C\langle x\rangle^{-\rho} \tag{6}$$

with $\rho > 0$,

$\rho > 1 \Longrightarrow$ the wave operators exist, (short-range)
$\rho \leqslant 1 \Longrightarrow$ the wave operators do not exist. (long-range)

That is to say, the borderline is $\rho = 1$.

The classical trajectory of the particle of the free Schrödinger has order

$$x(t) = O(t) \quad \text{as} \quad t \to \infty. \tag{7}$$

By substituting the classical order,

$$\begin{aligned}\int_1^\infty \|Ve^{-itH_0}\phi\|_{L^2} dt &= \int_1^\infty \|V(x(t))e^{-itH_0}\phi\|_{L^2} dt \\ &\leqslant C \int_1^\infty t^{-\rho} dt < \infty \iff \rho > 1.\end{aligned} \tag{8}$$

The case of the repulsive : $H_0 = p^2 - x^2$

By solving the Newton equation $\ddot{x}(t)/2 = 2x(t)$, the classical order is

$$x(t) = O(e^{2t}) \quad \text{as} \quad t \to \infty. \tag{9}$$

When we impose the decay condition on V by

$$|V(x)| \leqslant C(\log\langle x\rangle)^{-\rho}, \tag{10}$$

we can expect that the borderline will be

$$\rho = 1 \tag{11}$$

because of the analogy before.

Theorem[2]

Let the potential V be defined as

$$V(x) = \lambda(\log(2 + |x|))^{-\rho} \tag{12}$$

with $0 \neq \lambda \in \mathbb{R}$ and $0 < \rho \leqslant 1$.

For $H = H_0 + V$, the wave operators

$$\operatorname*{s-lim}_{t \to \pm\infty} e^{itH} e^{-itH_0} \tag{13}$$

do not exist.

[2] *J. Math. Anal. Appl.* **438** (2016)