Spectral decimation & its applications to spectral analysis on infinite fractal lattices

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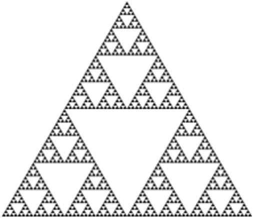
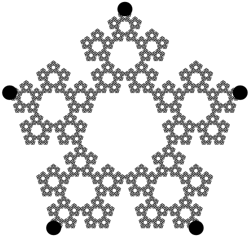
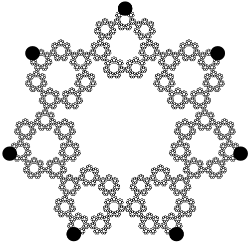
Joint works with S. Molchanov and A. Teplyaev

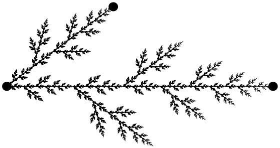
Colgate University

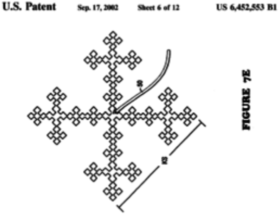
Motivation: Analysis on nonsmooth domains

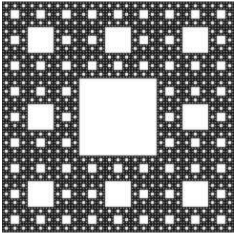




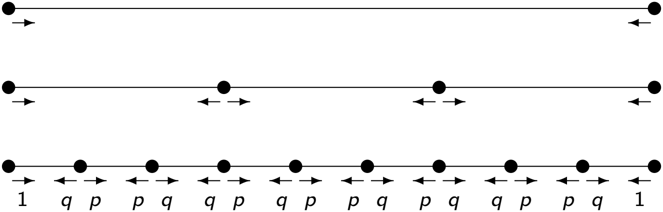


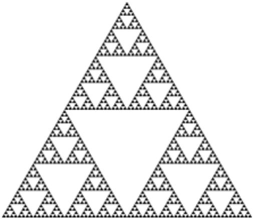


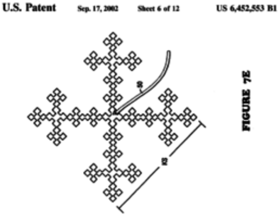




Some fractals are nicer than others

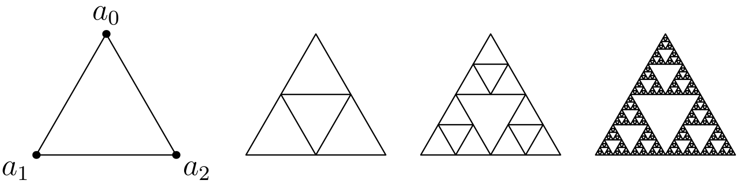






Each of these fractals is obtained from a nested sequence of graphs which has *nice, symmetric* replacement rules.

Spectral decimation (= spectral similarity)



Rammal-Toulouse ‘84, Bellissard ‘88, Fukushima-Shima ‘92, Shima ‘96, etc.
A recursive algorithm for identifying the Laplacian spectrum on highly symmetric, finitely ramified self-similar fractals.

Spectral decimation

Definition (Malozemov-Teplyaev '03)

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces. We say that an operator H on $\mathcal{H}$ is spectrally similar to H_0 on $\mathcal{H}_0$ with functions φ_0 and φ_1 if there exists a partial isometry $U : \mathcal{H}_0 \to \mathcal{H}$ (that is, $UU^* = I$) such that

$$U(H-z)^{-1}U^* = (\varphi_0(z)H_0 - \varphi_1(z))^{-1} =: \frac{1}{\varphi_0(z)}(H_0 - R(z))^{-1}$$

for any $z \in \mathbb{C}$ for which the two sides make sense.

A common class of examples: $\mathcal{H}_0$ subspace of $\mathcal{H}$, U^* is an ortho. projection from $\mathcal{H}$ to $\mathcal{H}_0$. Write $H - z$ in block matrix form w.r.t. $\mathcal{H}_0 \oplus \mathcal{H}_0^\perp$:

$$H - z = \begin{pmatrix} I_0 - z & \overline{X} \\ X & Q - z \end{pmatrix}.$$

Then $U(H-z)^{-1}U^*$ is the inverse of the Schur complement $S(z)$ w.r.t. to the lower-right block of $H - Z$: $S(z) = (I_0 - z) - \overline{X}(Q-z)^{-1}X$.

Issue: There may exist a set of z for which either $Q - z$ is not invertible, or $\varphi_0(z) = 0$.

Spectral decimation: the main theorem

Spectrum $\sigma(\Delta) = \{z \in \mathbb{C} : \Delta - z \text{ does not have a bounded inverse}\}$.

Definition

The exceptional set for spectral decimation is

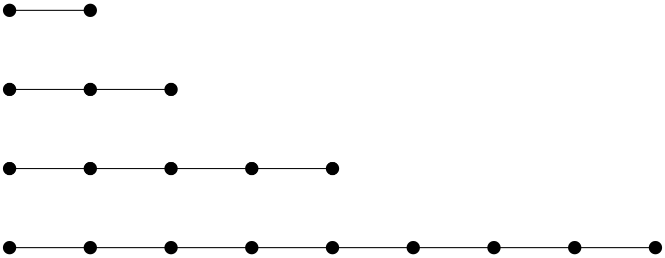
$$\mathfrak{E}(H, H_0) \stackrel{def}{=} \{z \in \mathbb{C} : z \in \sigma(Q) \text{ or } \varphi_0(z) = 0\}.$$

Theorem (Malozemov-Teplyaev '03)

Suppose H is spectrally similar to H_0. Then for any $z \notin \mathfrak{E}(H, H_0)$:

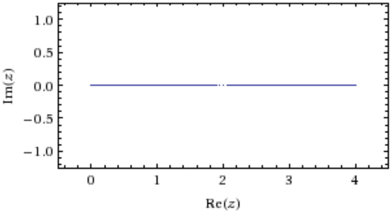
- $R(z) \in \sigma(H_0) \iff z \in \sigma(H)$.
- *$R(z)$ is an eigenvalue of H_0 iff z is an eigenvalue of H. Moreover there is a one-to-one map between the two eigenspaces.*

Example: $\mathbb{Z}_+$



Let Δ be the graph Laplacian on $\mathbb{Z}_+$ (with Neumann boundary condition at 0), realized as the limit of graph Laplacians on $[0, 2^n] \cap \mathbb{Z}_+$.
If $z \neq 2$ and $R(z) = z(4-z)$, then

- $R(z) \in \sigma(-\Delta) \Longleftrightarrow z \in \sigma(-\Delta)$.
- $\sigma(-\Delta) = \mathcal{J}_R$, where $\mathcal{J}_R$ is the Julia set of R.
- $\mathcal{J}_R$ is the full interval $[0, 4]$.



The *pq*-model

A one-parameter model of 1D fractals parametrized by $p \in (0,1)$. Set $q = 1 - p$.

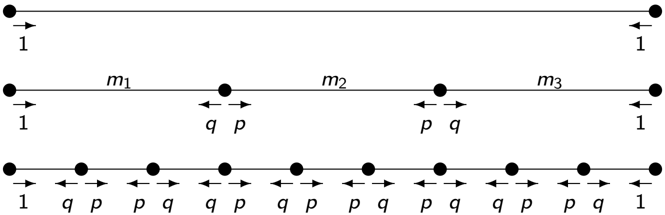
A triadic interval construction, "next easiest" fractal beyond the dyadic interval.

Earlier investigated by Kigami '04 (heat kernel estimates) and Teplyaev '05 (spectral decimation & spectral zeta function).

Assign probability weights to the three segments:

$$m_1 = m_3 = \frac{q}{1+q}, \quad m_2 = \frac{p}{1+q}$$

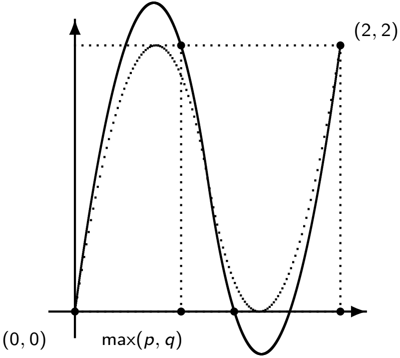
Then iterate. Let π be the resulting self-similar probability measure.



Spectral decimation for the pq-model

The spectral decimation polynomial is $R(z) = \frac{z(z^2 - 3z + (2+pq))}{pq}$.

$$\sigma(-\Delta_n) = \{0, 2\} \cup \bigcup_{m=0}^{n-1} R^{-m}\{1 \pm q\}$$



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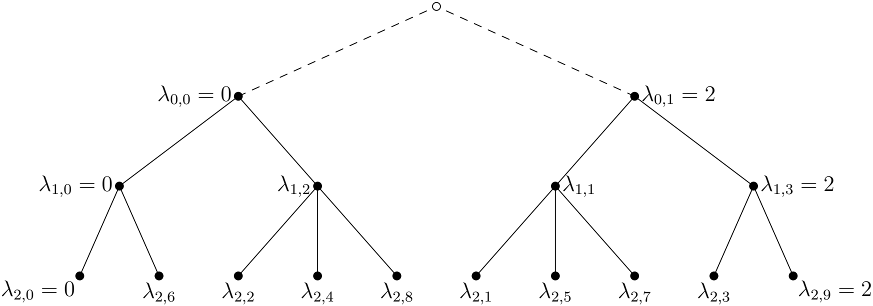
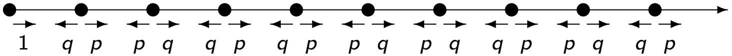


The pq-model on $\mathbb{Z}_+$

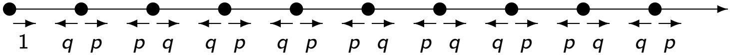


- Δ_p is not self-adjoint w.r.t. $\ell^2(\mathbb{Z}_+)$, but is self-adjoint w.r.t. the discretization of the aforementioned self-similar measure π.
- Let $\Delta_p^+ = D^* \Delta_p D$, where

$$D : \ell^2(\mathbb{Z}_+) \to \ell^2(3\mathbb{Z}_+), \quad (Df)(x) = f(3x).$$

Then Δ_p is spectrally similar to Δ_p^+. Moreover, Δ_p and Δ_p^+ are isometrically equivalent (in $L^2(\mathbb{Z}_+)$ or in $L^2(\mathbb{Z}_+, \pi)$).

The pq-model on $\mathbb{Z}_+$



Spectrum $\sigma(H) = \{z \in \mathbb{C} : H - z$ does not have a bounded inverse$\}$.
Facts from functional analysis:

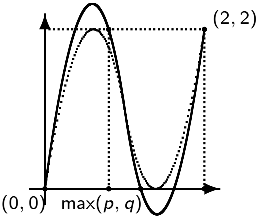
- $\sigma(H)$ is a nonempty compact subset of $\mathbb{C}$.
- $\sigma(H)$ equals the disjoint union $\sigma_{\rm pp}(H) \cup \sigma_{\rm ac}(H) \cup \sigma_{\rm sc}(H)$.
  pure point spectrum $\cup$ absolutely continuous spectrum $\cup$ singularly continuous spectrum

Theorem (C.-Teplyaev, *J. Math. Phys.* '16)

If $p \neq \frac{1}{2}$, the Laplacian Δ_p, regarded as an operator on either $\ell^2(\mathbb{Z}_+)$ or $L^2(\mathbb{Z}_+, \pi)$, has purely singularly continuous spectrum. The spectrum is the Julia set of the polynomial $R(z) = \frac{z(z^2 - 3z + (2+pq))}{pq}$, which is a topological Cantor set of Lebesgue measure zero.

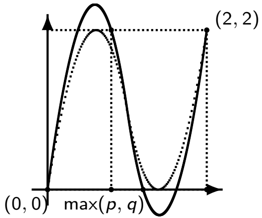
- One of the simplest realizations of purely singularly continuous spectrum. The mechanism appears to be simpler than those of quasi-periodic or aperiodic Schrodinger operators. (*cf.* Simon, Jitomirskaya, Avila, Damanik, Gorodetski, etc.)
- See also recent work of Grigorchuk-Lenz-Nagnibeda '14, '16 on spectra of Schreier graphs.

Proof of purely singularly continuous spectrum (when $p \neq \frac{1}{2}$)



1. Spectral decimation: Δ_p is spectrally similar to Δ_p^+, and they are isometrically equivalent. After taking into account the exceptional set, $R(z) \in \sigma(\Delta_p) \iff z \in \sigma(\Delta_p)$. Notably, the repelling fixed points of R, $\{0, 1, 2\}$, lie in $\sigma(\Delta_p)$.
2. By 1, $\overline{\bigcup_{n=0}^{\infty} R^{\circ -n}(0)} \subset \sigma(\Delta_p)$. Meanwhile, since $0 \in \mathcal{J}(R)$, $\overline{\bigcup_{n=0}^{\infty} R^{\circ -n}(0)} = \mathcal{J}(R)$. So $\mathcal{J}(R) \subset \sigma(\Delta_p)$.
3. If $z \in \sigma(\Delta_p)$, then by 1, $R^{\circ n}(z) \in \sigma(\Delta_p)$ for each $n \in \mathbb{N}$. On the one hand, $\sigma(\Delta_p)$ is compact. On the other hand, the only attracting fixed point of R is ∞, so $\mathcal{F}(R)$ (the Fatou set) contains the basin of attraction of ∞, whence non-compact. Infer that $z \notin \mathcal{F}(R) = (\mathcal{J}(R))^c$. So $\sigma(\Delta_p) \subset \mathcal{J}(R)$.

Proof of purely singularly continuous spectrum (when $p \neq \frac{1}{2}$)



4. Thus $\sigma(\Delta_p) = \mathcal{J}(R)$. When $p \neq \frac{1}{2}$, $\mathcal{J}(R)$ is a disconnected Cantor set. So $\sigma_{\rm ac}(\Delta_p) = \emptyset$.
5. Find the formal eigenfunctions corresponding to the fixed points of R, and show that none of them are in $\ell^2(\mathbb{Z}_+)$ and in $L^2(\mathbb{Z}_+, \pi)$. Thus none of the fixed points lie in $\sigma_{\rm pp}(\Delta_p)$. By spectral decimation, none of the pre-iterates of the fixed points under R are in $\sigma_{\rm pp}(\Delta_p)$. So $\sigma_{\rm pp}(\Delta_p) = \emptyset$.
6. Conclude that $\sigma(\Delta_p) = \sigma_{\rm sc}(\Delta_p)$.

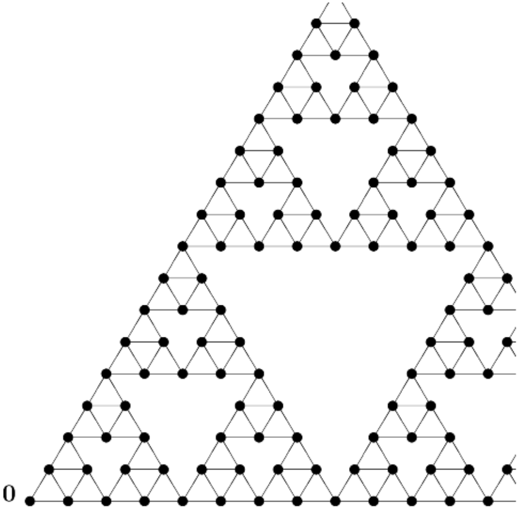
The Sierpinski gasket lattice (SGL)

Let Δ be the graph Laplacian on SGL.
If $z \notin \{2,5,6\}$ and $R(z) = z(5-z)$, then

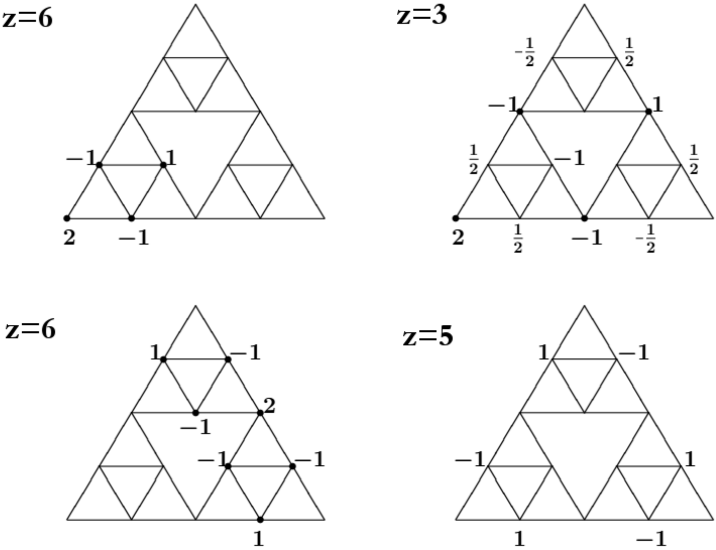
- $R(z) \in \sigma(-\Delta) \Longleftrightarrow z \in \sigma(-\Delta)$.
- $\sigma(-\Delta) = \mathcal{J}_R \cup \mathcal{D}$, where $\mathcal{J}_R$ is the Julia set of $R(z)$ and $\mathcal{D} := \{6\} \cup \left(\bigcup_{m=0}^{\infty} R^{-m}\{3\}\right)$.
- $\mathcal{J}_R$ is a disconnected Cantor set.

Thm. (Teplyaev '98)
On SGL, $\sigma(\Delta) = \sigma_{\mathrm{pp}}(\Delta)$.
Eigenfunctions with finite support are complete.

$\rightarrow$ Localization due to geometry.



Localized eigenfunctions on *SGL*



Random potential and Anderson localization

$H_\omega = -\Delta + V_\omega(x)$: ω denotes a realization of the random potential.

Definition (Anderson localization)

H_ω has spectral localization in an energy interval $[a,b]$ if, with probability 1, $\sigma(H_\omega)$ is p.p. in this interval. Furthermore, H_ω has exponential localization if the eigenfunctions with eigenvalues in $[a,b]$ decay exponentially.

Rigorous methods for proving (exponential) localization: Fröhlich-Spencer '83, Simon-Wolff '86, Aizenman-Molchanov '93.

Theorem (Aizenman-Molchanov '93, method of fractional moment of the resolvent)

Let $\tau(x,y;z) =: \mathbb{E}[\left|\langle x|(H_\omega - z)^{-1}|y\rangle\right|^s]$. If

$$\tau(x,y;E+i\epsilon) \leq Ae^{-\mu|x-y|}$$

for $E \in (a,b)$, uniformly in $\epsilon \neq 0$ and a suitable fixed $s \in (0,1)$, then H_ω has exponential localization.

The Aizenman-Molchanov estimate provides proofs of localization in the case of 1) large disorder, or 2) extreme energies.

Anderson localization on *SGL*

Theorem (Molchanov '16)

On SGL (and many other finitely ramified fractal lattices, $\sigma_{\rm ac}(H_\omega) = \emptyset$.

Proof. Based on the Simon-Wolff method.

Theorem (C.-Molchanov-Teplyaev '16+)

On SGL, the Aizenman-Molchanov estimate holds, i.e., for $E \in (a,b)$ *and* $E \notin \sigma(-\Delta)$,

$$\tau(x, y; E + i\epsilon) \leq Ae^{-\mu d(x,y)}$$

uniformly in $\epsilon \neq 0$ *and a suitable fixed* $s \in (0,1)$. *[*$d(\cdot,\cdot)$ *can be taken to be the graph metric.]*
As a consequence, H_ω *has exponential localization on SGL in the case of large disorder or extreme energies.*

Proof. If $E < 0$, then $\tau(x, y; E + i\epsilon)$ is a suitable Laplace transform of the heat kernel, which has a well-known sub-Gaussian upper estimate that decays exponentially with the graph distance $d(x, y)$:

$$\exists C_1, C_2 > 0 : \quad p_t(x,y) \leq C_1 t^{-\alpha} \exp\left(-\left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right) \qquad \forall x, y \in SGL, \quad \forall t > 0,$$

where $\alpha = \frac{\log 3}{\log 5}$, and $\beta = \frac{\log 5}{\log 2}$.

If $E > 0$, let $n(E)$ be the smallest natural number n such that $R^{\circ n}(E) < 0$, where $R(z) = z(5 - z)$. Use spectral decimation to relate the resolvent at E to the resolvent at $R^{\circ n}(E)$.

Thank you!