

Multivariate Trace Inequalities

Mario Berta

arXiv:1604.03023 with Sutter and Tomamichel (to appear in CMP)
arXiv:1512.02615 with Fawzi and Tomamichel

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Caltech

Motivation: Quantum Entropy

- **Entropy** of quantum states ρ_A on Hilbert spaces \mathcal{H}_A [von Neumann 1927]:

$$H(A)_\rho := -\text{tr} [\rho_A \log \rho_A] . \quad (1)$$

- **Strong subadditivity (SSA)** of tripartite quantum states on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ from matrix trace inequalities [Lieb & Ruskai 1973]:

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho . \quad (2)$$

- Generates **all known mathematical properties** of quantum entropy, manifold applications in quantum physics, quantum information theory, theoretical computer science etc.
- This talk: entropy for quantum systems, strengthening of SSA from **multivariate trace inequalities**.

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- Mark Wilde at 4pm: *Universal Recoverability in Quantum Information*.

- 1 Entropy for quantum systems
- 2 Multivariate trace inequalities
- 3 Proof of entropy inequalities
- 4 Conclusion

Entropy for classical systems

- **Entropy** of probability distribution P of random variable X over finite alphabet [Shannon 1948, Rényi 1961]:

$$H(X)_P := - \sum_x P(x) \log P(x), \quad \text{with } P(x) \log P(x) = 0 \text{ for } P(x) = 0. \quad (3)$$

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- Extension to **relative entropy** of P with respect to distribution Q over finite alphabet,

$$D(P\|Q) := \sum_x P(x) \log \frac{P(x)}{Q(x)} \quad [\text{Kullback \& Leibler 1951}], \quad (4)$$

where $P(x) \log \frac{P(x)}{Q(x)} = 0$ for $P(x) = 0$ and by continuity $+\infty$ if $P \not\ll Q$.

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- **Multipartite entropy measures** are generated through relative entropy, e.g., SSA:

$$H(XY)_P + H(YZ)_P \geq H(XYZ)_P + H(Y)_P \quad \text{equivalent to} \quad (5)$$

$$D(P_{XYZ}\|U_X \times P_{YZ}) \geq D(P_{XY}\|U_X \times P_Y) \quad \text{with } U_X \text{ uniform distribution.} \quad (6)$$

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- **Monotonicity of relative entropy (MONO)** under stochastic matrices N :

$$D(P\|Q) \geq D(N(P)\|N(Q)). \quad (7)$$

Entropy for quantum systems

- The **entropy** of $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is defined as:

$$H(A)_\rho := -\text{tr} [\rho_A \log \rho_A] = - \sum_x \lambda_x \log \lambda_x \quad [\text{von Neumann 1927}]. \quad (8)$$

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- **Commutative relative entropy** for $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ defined as

$$D_K(\rho||\sigma) := \sup_{\mathcal{M}} D(\mathcal{M}(\rho)||\mathcal{M}(\sigma)) \quad [\text{Donald 1986, Petz \& Hiai 1991}], \quad (9)$$

where $\mathcal{M} \in \text{CPTP}(\mathcal{H} \rightarrow \mathcal{H}')$ und $\text{Bild}(\mathcal{M}) \subseteq M \subseteq \text{Lin}(\mathcal{H}')$, M commutative subalgebra.

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- The **quantum relative entropy** is defined as

$$D(\rho \parallel \sigma) := \text{tr} [\rho (\log \rho - \log \sigma)] \quad [\text{Umegaki 1962}]. \quad (10)$$

- **Monotonicity (MONO)** for $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and $\mathcal{N} \in \text{CPTP}(\mathcal{H} \rightarrow \mathcal{H}')$:

$$D(\rho \parallel \sigma) \geq D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma)) \quad [\text{Lindblad 1975}]. \quad (11)$$

Entropy for quantum systems II

Theorem (Achievability of relative entropy, B. *et al.* 2015)

For $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\rho, \sigma > 0$ we have

$$D_K(\rho||\sigma) \leq D(\rho||\sigma) \quad \text{with equality if and only if } [\rho, \sigma] \neq 0. \quad (12)$$

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$$D(\rho||\sigma) = \sup_{\omega > 0} \text{tr} [\rho \log \omega] - \log \text{tr} [\exp(\log \sigma + \log \omega)] \quad [\text{Araki ?}, \text{Petz 1988}]. \quad (14)$$

- Golden-Thompson inequality:

$$\text{tr} [\exp(\log M_1 + \log M_2)] \leq \text{tr}[M_1 M_2]. \quad (15)$$

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- Proof: new matrix analysis technique **asymptotic spectral pinching** (see also [Hiai & Petz 1993, Mosonyi & Ogawa 2015]).

Asymptotic spectral pinching [B. *et al.* 2016]

- $A \geq 0$ with spectral decomposition $A = \sum_{\lambda} \lambda P_{\lambda}$, where $\lambda \in \text{spec}(A) \subseteq \mathbb{R}$ eigenvalues and P_{λ} orthogonal projections. **Spectral pinching** with respect to A defined as

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$$\begin{aligned} \text{(i)} \quad & [\mathcal{P}_A(X), A] = 0 \quad \text{(ii)} \quad \text{tr} [\mathcal{P}_A(X)A] = \text{tr} [XA] \quad \text{(iii)} \quad \mathcal{P}_A(X) \geq |\text{spec}(A)|^{-1} \cdot X \\ \text{(iv)} \quad & |\text{spec}(A \otimes \cdots \otimes A)| = |\text{spec}(A^{\otimes m})| \leq \mathcal{O}(\text{poly}(m)). \end{aligned} \quad (17)$$

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- Golden-Thompson inequality:

$$\log \text{tr} [\exp(\log A + \log B)] = \frac{1}{m} \log \text{tr} [\exp(\log A^{\otimes m} + \log B^{\otimes m})] \quad (18)$$

$$\leq \frac{1}{m} \log \text{tr} [\exp(\log A^{\otimes m} + \log \mathcal{P}_{A^{\otimes m}}(B^{\otimes m}))] + \frac{\log \text{poly}(m)}{m} \quad (19)$$

$$= \frac{1}{m} \log \text{tr} [A^{\otimes n} \mathcal{P}_{A^{\otimes m}}(B^{\otimes m})] + \frac{\log \text{poly}(m)}{m} \quad (20)$$

$$= \log \text{tr}[AB] + \frac{\log \text{poly}(m)}{m} \quad \square \quad (21)$$

Entropy for quantum systems III

- The right extension for applications is Umegaki's $D(\rho\|\sigma) = \text{tr} [\rho (\log \rho - \log \sigma)]$. Intuition chain rule [Petz 1992] with SSA:

$$H(AB)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(B)_\rho \quad \text{equivalent to} \quad (22)$$

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) \geq D(\rho_{AB}\|\tau_A \otimes \rho_B) \quad \text{with } \tau_A = \frac{1_A}{\dim(\mathcal{H}_A)}. \quad (23)$$

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$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \quad \Rightarrow \quad \text{strengthening of MONO/SSA?} \quad (24)$$

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- **Equality conditions** MONO [Petz 1986]:

Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ with $\rho \ll \sigma$ and $\mathcal{N} \in \text{CPTP}(\mathcal{H} \rightarrow \mathcal{H}')$. Then, we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = 0 \quad (25)$$

if and only if there exists $\mathcal{R}_{\sigma, \mathcal{N}} \in \text{CPTP}(\mathcal{H}' \rightarrow \mathcal{H})$ such that

$$\mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N}(\rho) = \rho \quad \text{und} \quad \mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N}(\sigma) = \sigma. \quad (26)$$

The quantum operation $\mathcal{R}_{\sigma, \mathcal{N}}$ is not unique, but can be chosen independent of ρ .

Strong monotonicity (sMONO)

Theorem (Strong monotonicity (sMONO), B. *et al.* 2016)

For the same premises as before, we have

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq D_K(\rho\|\mathcal{R}_{\sigma,\mathcal{N}} \circ \mathcal{N}(\rho)), \quad (27)$$

with

$$\mathcal{R}_{\sigma,\mathcal{N}}(\cdot) := \int_{-\infty}^{\infty} dt \beta_0(t) \sigma^{\frac{1+it}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-\frac{1+it}{2}} (\cdot) \mathcal{N}(\sigma)^{-\frac{1-it}{2}} \right) \sigma^{\frac{1-it}{2}} \in \text{CPTP}(\mathcal{H}' \rightarrow \mathcal{H})$$

$$\text{and } \beta_0(t) := \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}.$$

- Previous work: [Winter & Li 2012, Kim 2013, B. *et al.* 2015, Fawzi & Renner 2015, Wilde 2015, Junge *et al.* 2015, Sutter *et al.* 2016].

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- **Special case SSA (sSSA)**, becomes an equality in the commutative case:

$$D(\rho_{ABC}\|\tau_A \otimes \rho_{BC}) - D(\rho_{AB}\|\tau_A \otimes \rho_B) \geq D_K(\rho_{ABC}\|(\mathcal{I}_A \otimes \mathcal{R}_{B \rightarrow BC})(\rho_{AB})), \quad (28)$$

with $\mathcal{R}_{B \rightarrow BC}(\cdot) := \int_{-\infty}^{\infty} dt \beta_0(t) \rho_{BC}^{\frac{1+it}{2}} \left(\left(\rho_B^{-\frac{1+it}{2}} (\cdot) \rho_B^{-\frac{1-it}{2}} \right) \otimes 1_C \right) \rho_{BC}^{\frac{1-it}{2}}$, where $\mathcal{R}_{B \rightarrow BC} \in \text{CPTP}(\mathcal{H}_B \rightarrow \mathcal{H}_B \otimes \mathcal{H}_C)$.

Proof SSA

- Following [Lieb & Ruskai 1973] we have with Klein's inequality

$$D(\rho_{ABC} \| \tau_A \otimes \rho_{BC}) - D(\rho_{AB} \| \tau_A \otimes \rho_B) = D(\rho_{ABC} \| \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})) \quad (29)$$

$$\geq \text{tr}[\rho_{ABC} - \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})] \quad (30)$$

We could conclude SSA if $\text{tr}[\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})] \leq 1$.

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- Golden-Thompson $\text{tr}[\exp(\log M_1 + \log M_2)] \leq \text{tr}[M_1 M_2]$ to **Lieb's triple matrix inequality**:

$$\text{tr}[\exp(\log M_1 - \log M_2 + \log M_3)] \leq \int_0^\infty d\lambda \text{tr} [M_1 (M_2 + \lambda)^{-1} M_3 (M_2 + \lambda)^{-1}]$$

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- Idea: for sSSA start with the variational formula

$$\begin{aligned} & D(\rho_{ABC} \| \tau_A \otimes \rho_{BC}) - D(\rho_{AB} \| \tau_A \otimes \rho_B) \\ &= \sup_{\omega_{ABC} > 0} \text{tr}[\rho_{ABC} \log \omega_{ABC}] - \log \text{tr}[\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC} + \log \omega_{ABC})]. \end{aligned} \quad (32)$$

Multivariate trace inequalities

Theorem (Multivariate Golden-Thompson, B. *et al.* 2016)

Let $p \geq 1$, $n \in \mathbb{N}$, and $\{H_k\}_{k=1}^n$ be a set of hermitian matrices. Then, we have

$$\log \left\| \exp \left(\sum_{k=1}^n H_k \right) \right\|_p \leq \int_{-\infty}^{\infty} dt \beta_0(t) \log \left\| \prod_{k=1}^n \exp((1+it)H_k) \right\|_p, \quad (33)$$

where $\|M\|_p := \left(\text{tr} \left[(M^\dagger M)^{p/2} \right] \right)^{1/p}$ with $\beta_0(t) := \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}$.

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- Proof based on **Lie-Trotter expansion** $\exp(\sum_{k=1}^n H_k) = \lim_{r \rightarrow 0} \left(\prod_{k=1}^n \exp(rH_k) \right)^{1/r}$ extension of [Araki-Lieb-Thirring 1976/1990]:

Lemma (Multivariate Araki-Lieb-Thirring, B. et al. 2016)

Let $p \geq 1$, $r \in (0, 1]$, $n \in \mathbb{N}$, and $\{M_k\}_{k=1}^n$ be a set of positive matrices. Then, we have

$$\log \left\| \left\| \prod_{k=1}^n M_k^r \right\|^{1/r} \right\|_p \leq \int_{-\infty}^{\infty} dt \beta_r(t) \log \left\| \prod_{k=1}^n M_k^{1+it} \right\|_p, \quad (34)$$

with $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t) + \cos(\pi r))}$.

Complex interpolation theory

- Strengthening of **Hadamard's three line theorem** [Hirschman 1952]:

Let $S := \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1\}$, $g : S \rightarrow \mathbb{C}$ be uniformly bounded on S , holomorphic in the interior of S , and continuous on the boundary. Then, we have for $r \in (0, 1)$ with $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t) + \cos(\pi r))}$ that:

$$\log |g(r)| \leq \int_{-\infty}^{\infty} dt \beta_{1-r}(t) \log |g(it)|^{1-r} + \beta_r(t) \log |g(1+it)|^r \quad (35)$$

$$\leq \sup_t \log |g(it)|^{1-r} + \sup_t \log |g(1+it)|^r . \quad (36)$$

Complex interpolation theory

- Strengthening of **Hadamard's three line theorem** [Hirschman 1952]:

Let $S := \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1\}$, $g : S \rightarrow \mathbb{C}$ be uniformly bounded on S , holomorphic in the interior of S , and continuous on the boundary. Then, we have for $r \in (0, 1)$ with $\beta_r(t) := \frac{\sin(\pi r)}{2r(\cosh(\pi t) + \cos(\pi r))}$ that:

$$\log |g(r)| \leq \int_{-\infty}^{\infty} dt \beta_{1-r}(t) \log |g(it)|^{1-r} + \beta_r(t) \log |g(1+it)|^r \quad (35)$$

$$\leq \sup_t \log |g(it)|^{1-r} + \sup_t \log |g(1+it)|^r. \quad (36)$$

- Stein interpolation** for linear operators [Beigi 2013, Wilde 2015, Junge *et al.* 2015]:

Let $S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1\}$ and $G : S \rightarrow \operatorname{Lin}(\mathcal{H})$ be holomorphic in the interior of S and continuous on the boundary. For $p_0, p_1 \in [1, \infty]$, $r \in (0, 1)$, define p_r with $1/p_r = (1-r)/p_0 + r/p_1$. If $z \mapsto \|G(z)\|_{p_{\operatorname{Re}(z)}}$ is uniformly bounded on S , then we have for $\beta_r(t)$ as above:

$$\log \|G(r)\|_{p_r} \leq \int_{-\infty}^{\infty} dt \left(\beta_{1-r}(t) \log \|G(it)\|_{p_0}^{1-r} + \beta_r(t) \log \|G(1+it)\|_{p_1}^r \right). \quad (37)$$

Proof of multivariate trace inequalities

Lemma (Multivariate Araki-Lieb-Thirring, B. *et al.* 2016)

Let $p \geq 1$, $r \in (0, 1]$, $n \in \mathbb{N}$, and $\{M_k\}_{k=1}^n$ be a set of positive matrices. Then, we have

$$\log \left\| \left\| \prod_{k=1}^n M_k^r \right\|^{1/r} \right\|_p \leq \int_{-\infty}^{\infty} dt \beta_r(t) \log \left\| \prod_{k=1}^n M_k^{1+it} \right\|_p. \quad (38)$$

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■ Proof: Use Stein-Hirschman for $1/p_r = (1-r)/p_0 + r/p_1$:

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and choose

$$G(z) := \prod_{k=1}^n M_k^z = \prod_{k=1}^n \exp(z \log M_k) \quad \text{sowie} \quad p_0 := \infty, \quad p_1 := p, \quad p_r = \frac{p}{r}. \quad (40)$$

For positive matrices M_k , M_k^{it} becomes unitary, $\log \|\cdot\|_{p_0}^{1-r}$ in (39) becomes zero, and (38) follows. \square

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- **Multivariate Golden-Thompson** from Lie-Trotter expansion.

Proof of sSSA/sMONO

- The proof of sSSA follows from **multivariate Golden-Thompson for $p = 2$ and $n = 4$** :

$$\begin{aligned} & \text{tr} [\exp(\log M_1 - \log M_2 + \log M_3 + \log M_4)] \\ & \leq \int dt \beta_0(t) \text{tr} \left[M_1 M_2^{-(1+it)/2} M_3^{(1+it)/2} M_4 M_3^{(1-it)/2} M_2^{-(1-it)/2} \right]. \quad (41) \end{aligned}$$

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- Proof:** Choose $M_1 := \rho_{AB}$, $M_2 := \rho_B$, $M_3 := \rho_{BC}$, $M_4 := \omega_{ABC}$, and thus

$$D(\rho_{ABC} \| \tau_A \otimes \rho_{BC}) - D(\rho_{AB} \| \tau_A \otimes \rho_B) = D(\rho_{ABC} \| \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})) \quad (42)$$

$$= \sup_{\omega_{ABC} > 0} \operatorname{tr} [\rho_{ABC} \log \omega_{ABC}] - \log \operatorname{tr} [\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC} + \log \omega_{ABC})] \quad (43)$$

$$\geq \sup_{\omega_{ABC} > 0} \operatorname{tr} [\rho_{ABC} \log \omega_{ABC}] - \int dt \beta_0(t) \log \operatorname{tr} \left[\omega_{ABC} \rho_{BC}^{\frac{1+it}{2}} \rho_B^{-\frac{1+it}{2}} \rho_{AB} \rho_B^{-\frac{1-it}{2}} \rho_{BC}^{\frac{1+it}{2}} \right] \quad (44)$$

$$\geq D_K \left(\rho_{ABC} \| \int dt \beta_0(t) \rho_{BC}^{\frac{1+it}{2}} \rho_B^{-\frac{1+it}{2}} \rho_{AB} \rho_B^{-\frac{1-it}{2}} \rho_{BC}^{\frac{1+it}{2}} \right) \quad (45)$$

$$= D_K(\rho_{ABC} \| \mathcal{R}_{B \rightarrow BC}(\rho_{AB})) \quad \square \quad (46)$$

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$$\operatorname{tr}[M_1 \# M_2] \leq \operatorname{tr} [\exp(\log M_1 + \log M_2)] \leq \operatorname{tr}[M_1 M_2] \quad [\text{Hiai \& Petz 1993}]. \quad (47)$$

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- Mark Wilde at 4pm: *Universal Recoverability in Quantum Information*.