Microlocal methods for dynamical systems

QMath 13 Georgia Tech

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Dynamical systems

Dynamical systems

Dynamical systems: a statistical approach

Dynamical systems: a statistical approach

Linear

Non-linear

Dynamical systems: a statistical approach

Completely integrable Chaotic

In the chaotic case positions and directions get uniformly distributed:

Typical questions:

How long do we have to wait to have uniform distribution?

Are there periodic orbits and what information to they contain?

In the chaotic case positions and directions get uniformly distributed:

Recent work on the rate of decay for billiards by Baladi–Demers–Liverani '15

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That includes the time at which we achieve uniform distribution.

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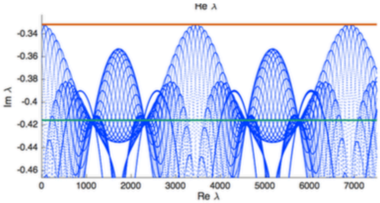
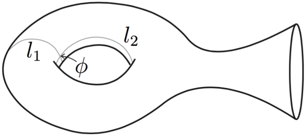
Selberg '56, Artin–Mazur '65, Smale '67, Bowen–Lanford '68, Ruelle '76, Milnor–Thurston '77, Parry–Pollicott '83,'90, Pollicott '86, Cvitanović–Eckhardt '91, Mayer, '91, Rugh '96, Fried '86, '95, Kitaev '99, Petkov–Stoyanov '07, Baladi–Tsujii '08, Stoyanov '11, Faure–Tsujii '13, Borthwick–Weich '15, ...





$Y(\ell_1, \ell_2, \phi)$ for $\ell_1 = \ell_2 = 7$, $\phi = \frac{\pi}{2}$

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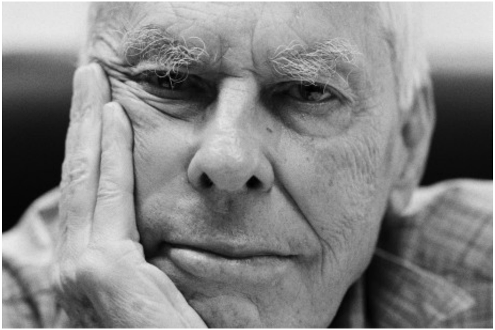
Smale '67 conjectured that for Anosov flows ζ_D is meromorphic in $\mathbb{C}$: "I must admit a positive answer would be a little shocking!"



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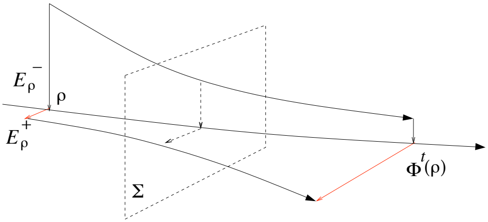

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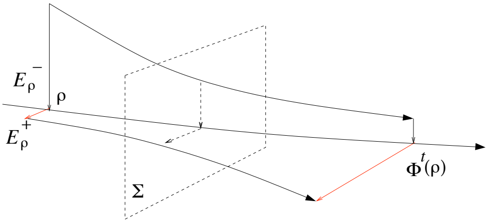

Example: $X = S^*M := \{(x,\xi) \in T^*M; |\xi|_g^2 = 1\}$, where (M,g) is a compact Riemannian manifold of negative curvature.

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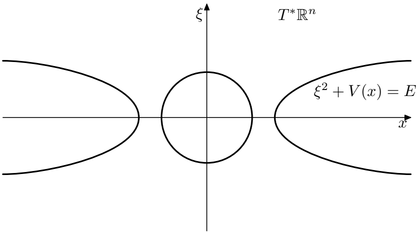
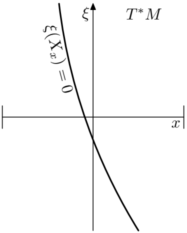
The noncompact case (essentially the full Smale conjecture) recently completed by Dyatlov–Guillarmou.



Scattering theory



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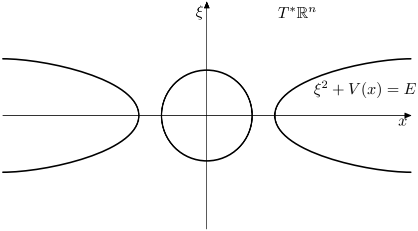

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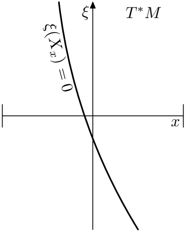

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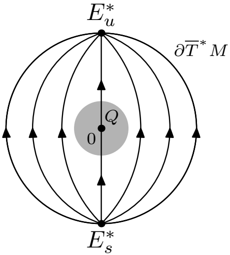
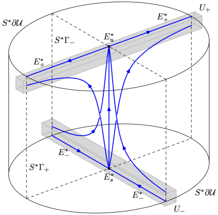


Dyatlov–Z



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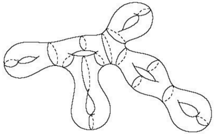
negative curvature. 

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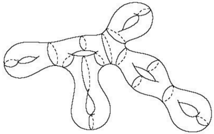

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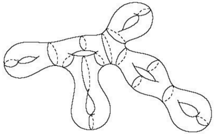

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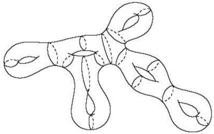

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In particular, the set of lengths of closed orbits (the length spectrum) determines the genus.

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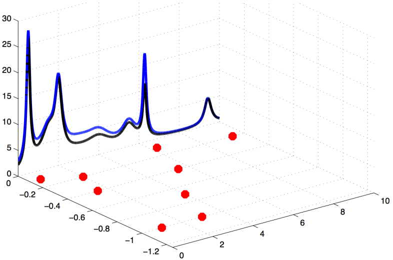

A “real” life example

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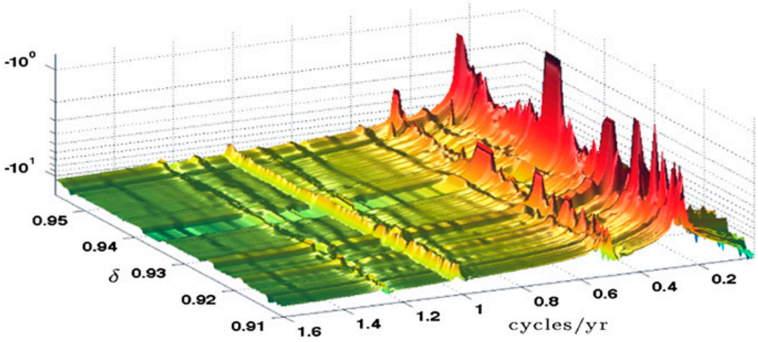

Rough parameter dependence in climate models and the role of Ruelle–Pollicott resonances,
Chekroun–Neelin–Kondrashov–McWilliams–Ghil, 2014

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$\nu_1 > 0$ for contact flows (and in particular for geodesic Anosov flows): Dolgopyat'98, Liverani '04, Tsujii'12, Nonnenmacher–Z'15.

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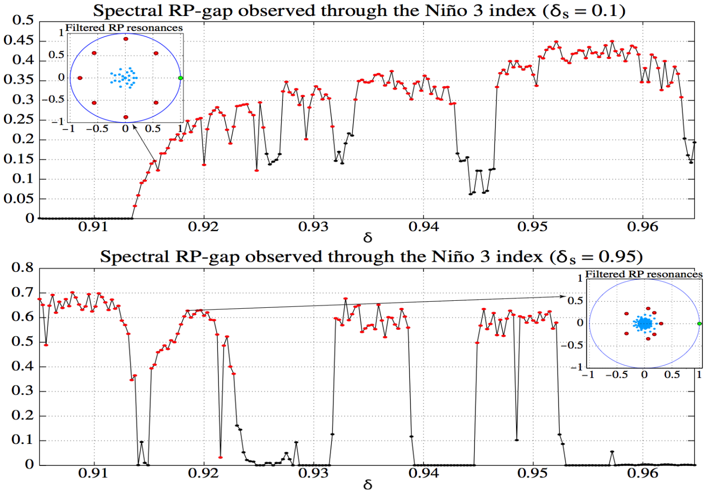

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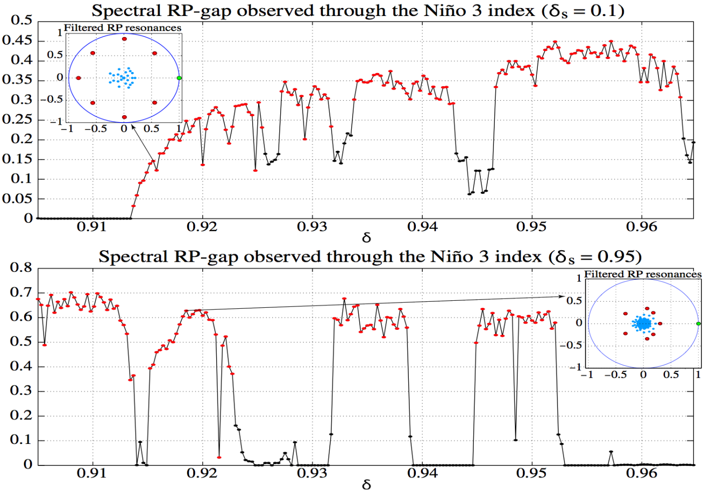

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Microlocal analysis (semiclassical version)

- Phase space: $(x, \xi) \in T^*X$
- Semiclassical parameter: $h \to 0$, the effective wavelength
- Classical observables: $a(x, \xi) \in C^\infty(T^*X)$
- Quantization: $\mathrm{Op}_h(a) = a\big(x, \frac{h}{i}\partial_x\big) : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$, semiclassical pseudodifferential operator

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Basic examples

- $a(x, \xi) = x_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = x_j$ multiplication operator
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Classical-quantum correspondence

- $[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\mathrm{Op}_h(\{a, b\}) + \mathcal{O}(h^2)$
- $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = H_a b, \quad e^{tH_a}$ Hamiltonian flow
- Example: $[\mathrm{Op}_h(\xi_k), \mathrm{Op}_h(x_j)] = \frac{h}{i}\delta_{jk}$

Standard semiclassical estimates

General question

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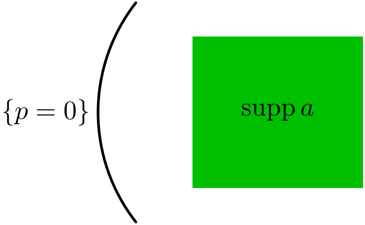

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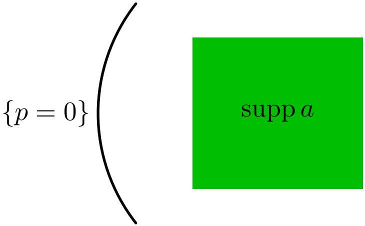
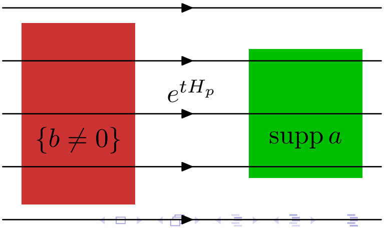


Propagation of singularities



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Examples:

- $\mathrm{WF}(\delta_0) = \{(0, \xi) : \xi \in \mathbb{R}^n \setminus 0\}$

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Phase space description of singularities

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Theorem (Dyatlov–Z '16)

$\zeta_k(\lambda)$ extends to an entire function and the multiplicities of its zeros are given by

$$\dim\left\{\mathbf{u} \in \mathcal{D}'(X, \Omega_0^k) \;:\; (\tfrac{1}{i}\mathcal{L}_V - \lambda)^r \mathbf{u} = 0,\ \mathrm{WF}(\mathbf{u}) \subset E_u^*\right\}$$

where Ω_0^k are k-forms satisfying $\iota_V \mathbf{u} = 0$.

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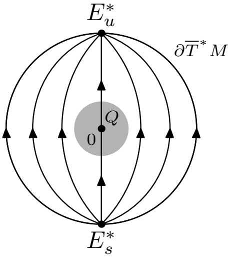

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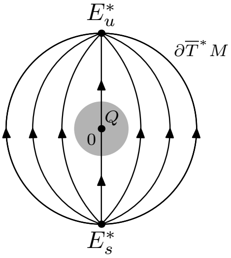

The meromorphy of $z \mapsto (P - z)^{-1} : \mathcal{C}^\infty(X) \to \mathcal{D}'(X)$ shows that the poles are simple and residues are positive integers.

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Here $L^2(X)$ is defined using the smooth invariant measure on S^*M:

$$d\mathrm{vol} := \alpha \wedge d\alpha$$

where α is the contact form,

$$\alpha = z d\zeta|_{S^*M}, \quad S^*M := \{(z, \zeta) \in T^*M : |\zeta|^2_{g(z)} = 1\},$$

$$\alpha(V) = 1, \quad \iota_V d\alpha = 0, \quad \ker \alpha(x) = E_u(x) \oplus E_s(x)$$

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But $\int_X Vu d\mathrm{vol} = 0$ so $\mathrm{const} = 0$.

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1. $\mathcal{L}_V \mathbf{u} = 0$ is equivalent to (since $\iota_V \mathbf{u} = 0$) to $\iota_V d\mathbf{u} = 0$. Hence $d\mathbf{u}$ is a resonant state for 2-forms and $d\mathbf{u} = c d\alpha$. Since $\mathbf{u} \wedge d\alpha = \iota_V \mathbf{u} \alpha \wedge d\alpha = 0$ we have

$$c \mathrm{vol}(X) = \int d\mathbf{u} \wedge \alpha = \int \mathbf{u} \wedge d\alpha = 0.$$

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Showing semisimplity is a little bit tricky...

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Since for surfaces of genus $g \geq 2$,

$$H^1(S^*M, \mathbb{C}) \simeq H^1(M, \mathbb{C})$$

it follows that the order of vanishing of ζ_D at 0 is

$$m = -m_0(0) + m_1(0) - m_2(0) = -\text{Euler characteristic of } M = 2g - 2$$

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Thanks for your attention!