Quasiperiodic Schrodinger operators: sharp arithmetic spectral transitions and universal hierarchical structure of eigenfunctions

S. Jitomirskaya

Atlanta, October 10, 2016

Almost Mathieu operators

$$(H_{\lambda,\alpha,\theta}\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + \lambda v(\theta + n\alpha)\Psi_n$$

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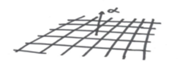

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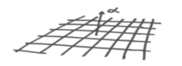

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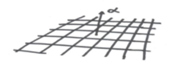

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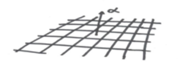

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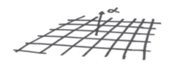

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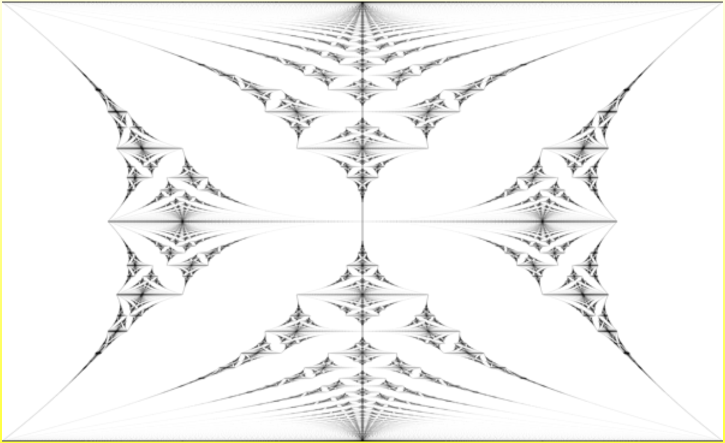
- With a choice of Landau gauge effectively reduces to h_θ
- α is a dimensionless parameter equal to the ratio of flux through a lattice cell to one flux quantum.

Hofstadter butterfly



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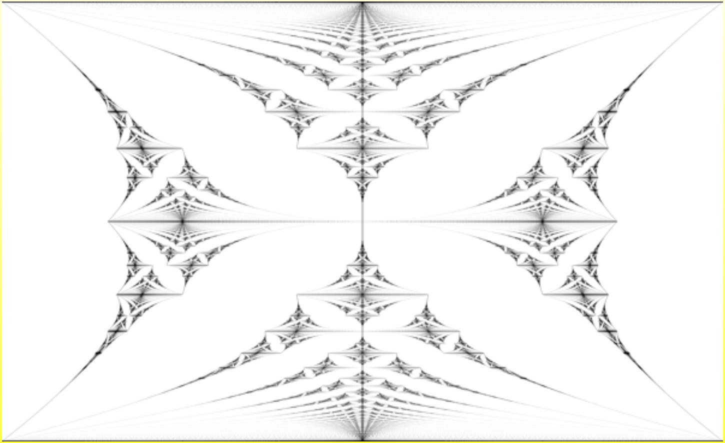

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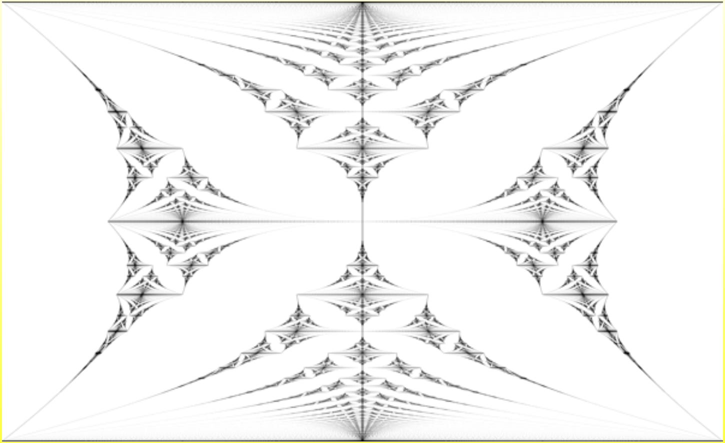

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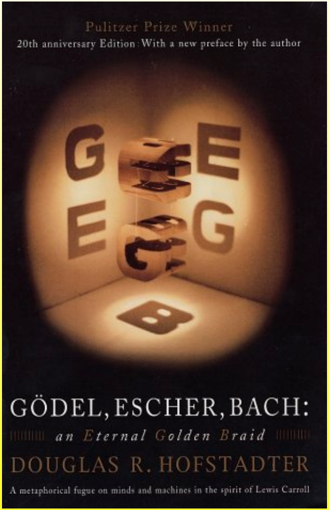
David Jennings described it as a picture of God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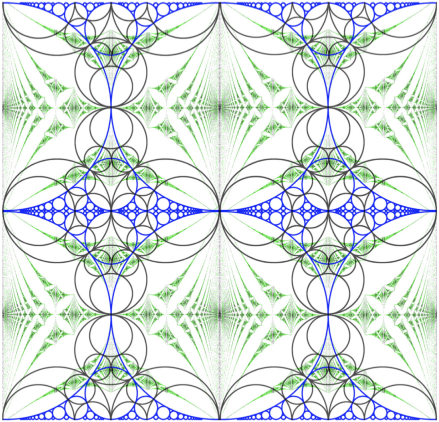
20th anniversary Edition With a new preface by the author
GÖDEL, ESCHER, BACH:
an Eternal Golden Braid
DOUGLAS R. HOFSTADTER
A metaphorical fugue on minds and machines in the spirit of Lewis Carroll

Butterfly in the Quantum World

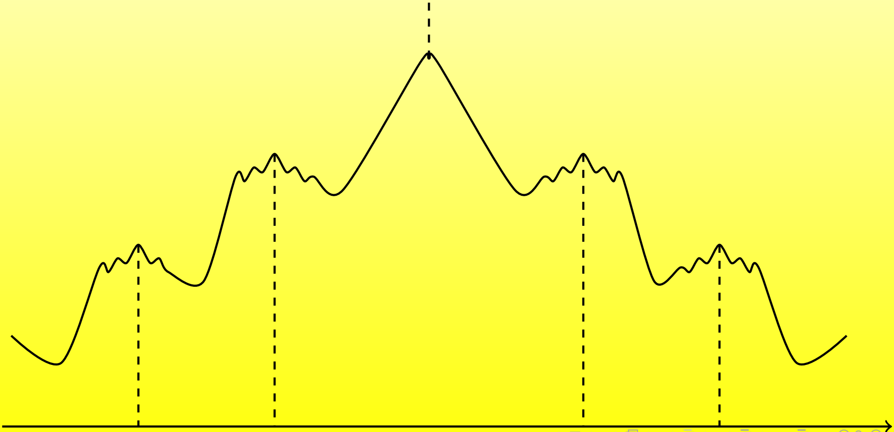
The story of the most fascinating quantum fractal

Indubala I Satija

with contributions by Douglas Hofstadter



Hierarchical structure driven by the continued fraction expansion of the magnetic flux: eigenfunctions



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Today: universal self-similar exponential structure of eigenfunctions throughout the entire localization regime.

Arithmetic spectral transitions

1D Quasiperiodic operators:

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- originally approached by KAM (Dinaburg, Sinai, Bellissard, Frohlich-Spencer-Wittwer, Eliasson)
- nonperturbative methods (SJ, Bourgain-Goldstein for $L > 0$; Last,SJ,Avila for $L = 0$) reduced the transition to the transition in the Lyapunov exponent (for analytic v):
  $L(E) > 0$ implies pp spectrum for a.e. α, θ
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$$\begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = A_n^E(\alpha,\theta) \begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix}$$
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Arithmetic transitions in the supercritical ($L > 0$) regime

Small denominators - resonances - $(v(\theta + k\alpha) - v(\theta + \ell\alpha))^{-1}$ are in competition with $e^{L(E)|\ell - k|}$.

L very large compared to the resonance strength leads to more localization

L small compared to the resonance strength leads to delocalization

Pure point to singular continuous transition conjecture

Exponential strength of a resonance:

$$\beta(\alpha) := \limsup_{n\to\infty} -\frac{\ln ||n\alpha||_{\mathbb{R}/\mathbb{Z}}}{|n|}$$

and

$$\delta(\alpha,\theta) := \limsup_{n\to\infty} -\frac{\ln ||2\theta + n\alpha||_{\mathbb{R}/\mathbb{Z}}}{|n|}$$

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$\lambda > 1 \to$ no ac spectrum (Ishii-Kotani-Pastur)

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Conjecture for the sharp transition (1994):

- If $\beta(\alpha) = 0$, then $\lambda_0 = e^{\delta(\alpha,\theta)}$ is the transition line:
  - $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for $|\lambda| < e^{\delta(\alpha,\theta)}$,
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(SJ-W.Liu, 16)
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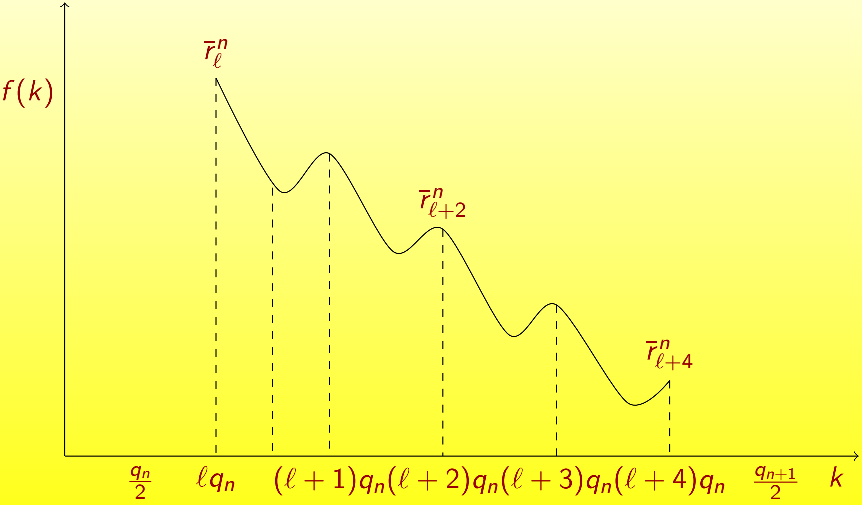
Note: $f(k)$ decays exponentially and $g(k)$ grows exponentially.
However the decay rate and growth rate are not always the same.

The behavior of $f(k)$



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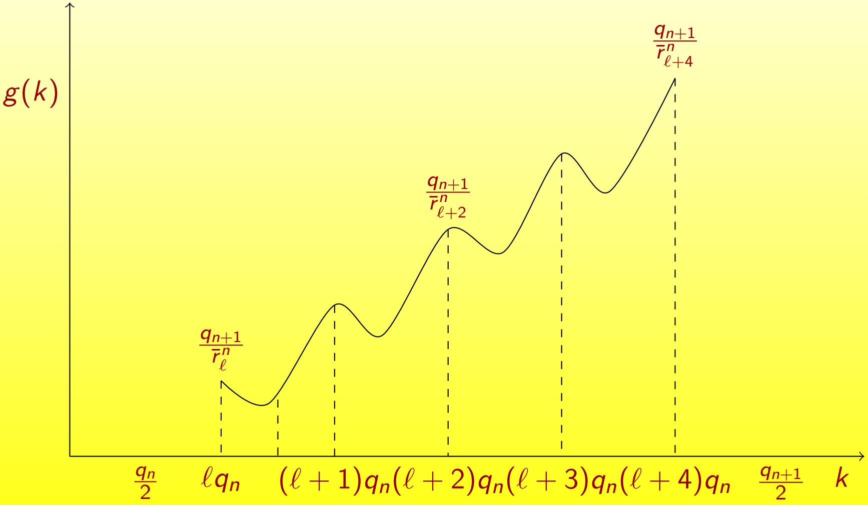

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Corollary

The arithmetic spectral transition conjecture holds as stated.

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Localization Method:

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- Avila-You-Zhou proved that there exists a full Lebesgue measure set S such that for $\theta \in S$, $H_{\lambda,\alpha,\theta}$ satisfies AL if $|\lambda| > e^{\beta(\alpha)}$, thus proving the transition line at $|\lambda| > e^{\beta(\alpha)}$ for a.e. θ. However, S can not be described in their proof.

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Local j-maxima

Local j-maximum is a local maximum on a segment $|I| \sim q_j$.
A local j-maximum k_0 is *nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{{q_{j-1}}^{\nu}},$$

for all $|k| \leq 2q_{j-1}$ and

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{0.1}$$

for all $2q_{j-1} < |k| \leq 2q_j$.
A local j-maximum is *strongly nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{0.2}$$

for all $0 < |k| \leq 2q_j$.

Universality of behavior at all (strongly) nonresonant local maxima:

Theorem

(SJ-W.Liu, 16) Suppose k_0 *is a local* j*-maximum. If* k_0 *is nonresonant, then*

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{0.3}$$

for all $2s \in I$, $|s| > \frac{q_{j-1}}{2}$.
If k_0 *is strongly nonresonant, then*

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{0.4}$$

for all $2s \in I$.

Universal hierarchical structure

All α, Diophantine θ, pp regime. Let k_0 be the global maximum

Theorem

(SJ-W. Liu, 16) There exists $\hat{n}_0(\alpha,\lambda,\varsigma,\epsilon)<\infty$ *such that for any* $k\geq\hat{n}_0$, $n_{j-k}\geq\hat{n}_0+k$, *and* $0<a_{n_i}<e^{\varsigma\ln|\lambda|q_{n_i}}$, $i=j-k,\ldots,j$, *for all* $0\leq s\leq k$ *there exists a local* n_{j-s}*-maximum* $b_{a_{n_j},a_{n_{j-1}},\ldots,a_{n_{j-s}}}$ *such that the following holds:*

I. $|b_{a_{n_j}}-(k_0+a_{n_j}q_{n_j})|\leq q_{\hat{n}_0+1}$,

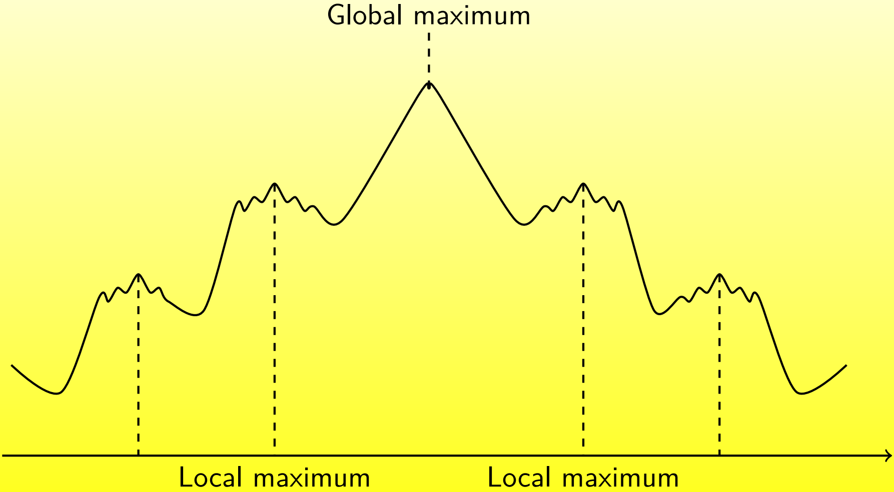
II.For $s\leq k$, $|b_{a_{n_j},\ldots,a_{n_{j-s}}}-(b_{a_{n_j},\ldots,a_{n_{j-s+1}}}+a_{n_{j-s}}q_{n_{j-s}})|\leq q_{\hat{n}_0+s+1}$.

III. if $q_{\hat{n}_0+k}\leq|(x-b_{a_{n_j},a_{n_{j-1}},\ldots,a_{n_{j-k}}}|\leq cq_{n_{j-k}}$, *then for* $s=0,1,\ldots,k$,

$$f(x_s)e^{-\varepsilon|x_s|}\leq\frac{||U(x)||}{||U(b_{a_{n_j},a_{n_{j-1}},\ldots,a_{n_{j-s}}})||}\leq f(x_s)e^{\varepsilon|x_s|},$$

Moreover, every local n_{j-s}*-maximum on the interval* $b_{a_{n_j},a_{n_{j-1}},\ldots,a_{n_{j-s+1}}}+[-e^{\epsilon\ln\lambda q_{n_{j-s}}},e^{\epsilon\ln\lambda q_{n_{j-s}}}]$ *is of the form* $b_{a_{n_j},a_{n_{j-1}},\ldots,a_{n_{j-s}}}$ *for some* $a_{n_{j-s}}$.

Universal hierarchical structure of the eigenfunctions



Universal reflexive-hierarchical structure

Theorem

(SJ-W. Liu,16) For Diophantine α and all θ in the pure point regime there exists a hierarchical structure of local maxima as above, such that

$$f((-1)^{s+1}x_s)e^{-\varepsilon|x_s|} \leq \frac{||U(x_s)||}{||U(b_{K_j,K_{j-1},\ldots,K_{j-s}})||} \leq f((-1)^{s+1}x_s)e^{\varepsilon|x_s|},$$

where $x_s = x - b_{K_j,K_{j-1},\ldots,K_{j-s}}$.

Further corollaries

Corollary

Let $\psi(k)$ be any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ that is linearly independent with respect to $\phi(k)$. Let $\bar{U}(k) = \begin{pmatrix} \psi(k) \\ \psi(k-1) \end{pmatrix}$, then

$$g(|k|)e^{-\varepsilon|k|} \leq ||\bar{U}(k)|| \leq g(|k|)e^{\varepsilon|k|}.$$

Let $0 \leq \delta_k \leq \frac{\pi}{2}$ be the angle between vectors $U(k)$ and $\bar{U}(k)$.

Corollary

We have

$$\limsup_{k\to\infty} \frac{\ln \delta_k}{k} = 0,$$

and

$$\liminf_{k\to\infty} \frac{\ln \delta_k}{k} = -\beta.$$

Corollary

We have

i)

$$\limsup_{k\to\infty} \frac{\ln ||A_k||}{k} = \limsup_{k\to\infty} \frac{\ln ||\bar{U}(k)||}{k} = \ln |\lambda|,$$

ii)

$$\liminf_{k\to\infty} \frac{\ln ||A_k||}{k} = \liminf_{k\to\infty} \frac{\ln ||\bar{U}(k)||}{k} = \ln |\lambda| - \beta.$$

iii) *Outside an explicit sequence of lower density zero,*

$$\lim_{k\to\infty} \frac{\ln ||A_k||}{k} = \lim_{k\to\infty} \frac{\ln ||\bar{U}(k)||}{k} = \ln |\lambda|.$$

Corollary

We have

i) $\limsup_{k\to\infty} \frac{-\ln \|U(k)\|}{k} = \ln|\lambda|$,

ii) $\liminf_{k\to\infty} \frac{-\ln \|U(k)\|}{k} = \ln|\lambda| - \beta$.

iii) *There is an explicit sequence of upper density* $1 - \frac{1}{2}\frac{\beta}{\ln|\lambda|}$, *along which*

$$\lim_{k\to\infty} \frac{-\ln \|U(k)\|}{k} = \ln|\lambda|.$$

iv) *There is an explicit sequence of upper density* $\frac{1}{2}\frac{\beta}{\ln|\lambda|}$,*along which*

$$\limsup_{k\to\infty} \frac{-\ln \|U(k)\|}{k} < \ln|\lambda|.$$

Further applications

- Upper bounds on fractal dimensions of spectral measures and quantum dynamics for trigonometric polynomials (SJ-W.Liu-S.Tcheremchantzev, SJ-W.Liu).
- The exact rate for exponential dynamical localization in expectation for the Diophantine case (SJ-H.Krüger-W.Liu). The first result of its kind, for any model.
- The same universal asymptotics of eigenfunctions for the Maryland Model (R. Han-SJ-F.Yang).

Key ideas of the proof

Resonant points (small divisors): $k : ||k\alpha||_{\mathbb{R}/\mathbb{Z}}$ or $||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}$ is small.

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- New way to deal with resonant points in the positive Lyapunov regime (supercritical regime)

Key ideas of the proof

Resonant points (small divisors): $k : ||k\alpha||_{\mathbb{R}/\mathbb{Z}}$ or $||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}$ is small.

- New way to deal with resonant points in the positive Lyapunov regime (supercritical regime)
- Develop Gordon and palindromic methods to study the trace of transfer matrices to obtain lower bounds on solutions
  Gordon potential (periodicity): $|V(j+q_n) - V(j)|$ is small (control by $||q_n\alpha|| \simeq e^{-\beta(\alpha)q_n}$)
  palindromic potential (symmetry): $|V(k-j) - V(j)|$ is small (control by $||2\theta + k\alpha|| \simeq e^{-\delta(\alpha,\theta)|k|}$)
- Bootstrap starting around the (local) maxima leads to effective estimates
- Reverse induction proof that local $j-1$-maxima are close to aq_{j-1} shifts of the local j-maxima, up to a constant scale
- Deduce that all the local maxima are (strongly) non-resonant and apply reverse induction

Assume E is a generalized eigenvalue and ϕ is the associated generalized eigenfunction $(|\phi(n)| < 1 + |n|)$. Let φ be another solution of $Hu = Eu$. Let $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$ and $\bar{U}(k) = \begin{pmatrix} \varphi(k) \\ \varphi(k-1) \end{pmatrix}$.

Step 1: Sharp estimates for the non-resonant points.

- $||U(k)|| \simeq e^{-\ln\lambda|k-k_i|}||U(k_i)|| + e^{-\ln\lambda|k-k_{i+1}|}||U(k_{i+1})||$
- $||\bar{U}(k)|| \simeq e^{-\ln\lambda|k-k_i|}||\bar{U}(k_i)|| + e^{-\ln\lambda|k-k_{i+1}|}||\bar{U}(k_{i+1})||$

where k_i is the resonant point and $k \in [k_i, k_{i+1}]$.

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Step 2: Sharp estimates for the resonant points.

- $||U(k_{i+1})|| \simeq e^{-c(k_i,k_{i+1})|k_{i+1}-k_i|}||U(k_i)||$
- $||\bar{U}(k_{i+1})|| \simeq e^{c'(k_i,k_{i+1})|k_{i+1}-k_i|}||\bar{U}(k_i)||$

where $c(k_i, k_{i+1}), c'(k_i, k_{i+1})$ can be given explicitly.

Current quasiperiodic preprints

Almost Mathieu operator:

- Avila-You-Zhou: sharp transition in α between pp and sc
- Avila-You-Zhou: dry Ten Martini, non-critical, all α
- Shamis-Last, Krasovsky, SJ- S. Zhang: gap size/dimension results for the critical case
- Avila-SJ-Zhou: critical line $\lambda = e^{\beta}$
- Damanik-Goldstein-Schlag-Voda: homogeneous spectrum, Diophantine α
- W. Liu-SJ: sharp transitions in α and θ and universal (reflective) hierarchical structure

Unitary almost Mathieu:

Fillman-Ong-Z. Zhang: complete a.e. spectral description

Current quasiperiodic preprints

Extended Harper's model:

- Avila-SJ-Marx: complete spectral description in the coupling phase space (+Erdos-Szekeres conjecture!)
- R. Han: an alternative argument
- R. Han-J: sharp transition in α between pp and sc spectrum in the positive Lyapunov exponent regime
- R. Han: dry Ten Martini (non-critical Diophantine)

General 1-frequency quasiperiodic:

analytic: SJ- S. Zhang: sharp arithmetic criterion for full spectral dimensionality (quasiballistic motion)

R. Han-SJ: sharp topological criterion for dual reducibility to imply localization

Damanik-Goldstein-Schlag-Voda: homogeneous spectrum, supercritical

monotone: SJ-Kachkovskiy: *all* coupling localization

meromorphic: SJ-Yang: sharp criterion for sc spectrum

Current quasiperiodic preprints

Maryland model:
W. Liu-SJ: complete arithmetic spectral transitions for *all* λ, α, θ
W. Liu: surface Maryland model
SJ-Yang: a constructive proof of localization
General Multi-frequency:

- R. Han-SJ: localization-type results with arithmetic conditions (general zero entropy dynamics; including the skew shift)
- R. Han-Yang: generic continuous spectrum
- Hou-Wang-Zhou: ac spectrum for Liouville (presence)
- Avila-SJ: ac spectrum for Liouville (absence)

Deift's problem (almost periodicity of KdV solutions with almost periodic initial data) :
Binder-Damanik-Goldstein-Lukic: a solution under certain conditions.