Robust Bounds for Emptiness Formation Probability for Dimers

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Based on joint work with Scott Williams, UAB (undergraduate).

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Outline for talk

- Review Emptiness Formation Probability for the XXZ Chain.
- Lieb, Sutherland, Baxter relation to the 6-vertex model.
- Reflection positivity.
- Dimer model relation to the 6-vertex model.
- One-sided bound for the dimer model, and open problems.

Emptiness Formation Probability in the XXZ model

Let $\Lambda(N) = \mathbb{Z}/N\mathbb{Z}$ be $\{1, \ldots, N\}$ with pbc $(N + 1 \equiv 1)$

$$H_{\Lambda(N)}(\Delta) = \sum_{k=1}^{N} (S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + \Delta S_k^z S_{k+1}^z) .$$

We restrict attention to even N.
Using the unitary $U_N = \exp(i\pi S_{2k}^z) \exp(i\pi S_4^z) \cdots \exp(i\pi S_N^z)$,

$$\begin{aligned} U_N H_{\Lambda(N)}(\Delta) U_N^* &= -\sum_{k=1}^{N} (S_k^x S_{k+1}^x + S_k^y S_{k+1}^y - \Delta S_k^z S_{k+1}^z) \\ &= -\sum_{k=1}^{N} \left(\frac{1}{2} S_k^+ S_{k+1}^- + \frac{1}{2} S_k^- S_{k+1}^+ - \Delta S_k^z S_{k+1}^z \right) . \end{aligned}$$

Restricted to 0 magnetization, there is a unique energy minimizer.

For $L \leq N$, the EFP projector is

$$Q_L = \prod_{k=1}^{L} \left(\frac{1}{2} + S_k^z \right) .$$

Korepin, Lukyanov, Nishiyama and Shiroishi (Phys.Lett.A, 2003) argued, algebraically,

$$\lim_{N\to\infty} \lim_{\beta\to\infty} \frac{\mathrm{tr}[Q_L e^{-\beta H_N^{\Delta}}]}{\mathrm{tr}[e^{-\beta H_N^{\Delta}}]} \sim A L^{-\gamma} C^{-L^2} ,$$

for C and γ which are explicit in $\Delta \in (-1, 1]$.

With Nick Crawford and Stephen Ng (CMP, 2016) we proved

Theorem. *Suppose* $\Delta > -1$ *(where* $\Delta = -1$ *is the* *ferromagnet).*

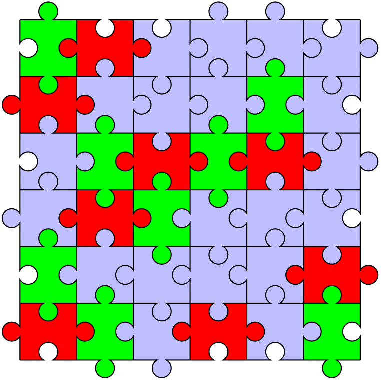
Let $\langle \cdots \rangle_{N,\Delta}^{\mathrm{GS}} =$ *the ground state expectation*
for H_N^{Δ} *with magnetization* 0*.*

Then there exist constants $c_1, C_1, c_2, C_2 \in (0, \infty)$ *with*

$$C_1 e^{-c_1 L^2} \leq \langle Q_L \rangle_{N,\Delta}^{\mathrm{GS}} \leq C_2 e^{-c_2 L^2},$$

for all even N *with* $L \leq N/2$*.*

6-vertex configurations = alternating sign matrices



$$= \begin{bmatrix} -1 & +1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & +1 & -1 & +1 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & +1 \\ +1 & -1 & 0 & +1 & 0 & -1 \end{bmatrix}$$

In 1970 Sutherland showed that the row-to-row transfer matrix of the 6-vertex model commutes with the XXZ model

[16] B. Sutherland. Two-Dimensional Hydrogen Bonded Crystals without the Ice Rule. *J. Math. Phys.* **11**, no. 11, 3183–3186 (1970).

Weight of a vertex is e^{κ} if it is a sink/source, 1 otherwise.
Let $A_{N,\kappa}$ be the row-to-row transfer matrix.
Then $A_{N,\kappa}$ and H_N^{Δ} commute if

$$\Delta = \frac{1}{2} e^{2\kappa} - 1 .$$

Note $\{\kappa \in (-\infty, \infty)\} \Leftrightarrow \{\Delta \in (-1, \infty)\}$.

[13] E. H. Lieb. Exact solution of the F model of an antiferroelectric. *Phys. Rev. Lett.* **18**, no. 24, 1046–1048 (1967).

In 1967 Lieb showed that the ground state for the XXZ model is the same as the invariant measure for the 6-vertex model.
Need #rows $\to \infty$.

Reflection Positivity of the Six Vertex Model

In 1980, Fröhlich, Israel, Lieb and Simon proved that the 6-vertex model is reflection positive.

[4] J. Frohlich, R. Israel, E. H. Lieb and B. Simon. Phase Transitions and Reflection Positivity. II. Lattice Systems with Short-Range and Coulomb Interactions. *J. Statist. Phys.* **22**, no. 3, 297–347 (1980).

They actually showed that the 6-vertex model is reflection positive for 2 types of reflections:

- coordinate directions, and
- diagonal directions.

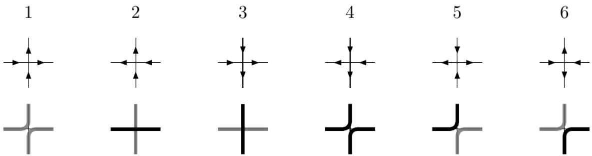


Figure 8: This is a translation of the 6 valid vertex configurations, and 6 vertex configuriations for the osculating paths. We enumerate them for later reference.

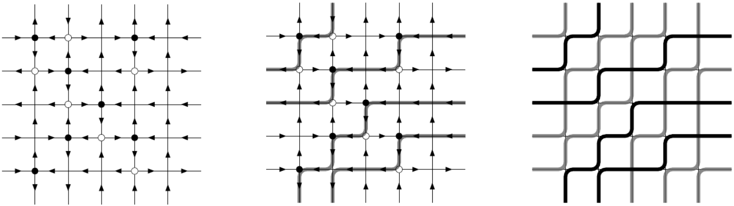


Figure 9: This picture is an example of a valid 6-vertex configuration and its mapping to an osculating path configuration (opc). (The middle picture is an intermediate step to guide the eye.)

R.J.Baxter (Ann.Phys. 1973) showed how to embed the dimer problem into the family of 8 vertex models on a diagonal lattice:

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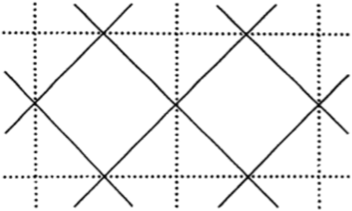
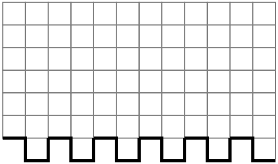
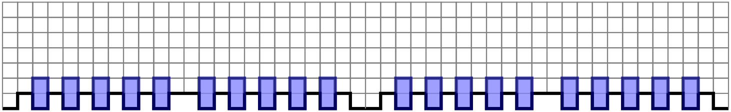


FIG. 3. The lattice transformation of Appendix A. A dimer covering of the original lattice (solid lines) is equivalent to an eight-vertex model on the superbond lattice (dotted lines).

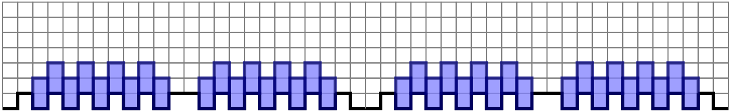
The EFP event defined on the set of dimers is that on a particular row, in a particular sub-interval, we see this



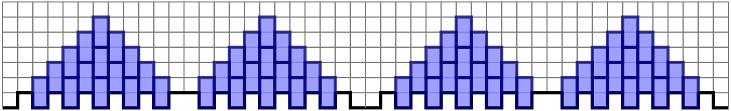
Under reflections, this looks like this:



This would force at least a frozen region for a while. For example, the next row is this:



And we can continue in this fashion.



Using this and reflection positivity it is easy to derive upper bounds similar to those before.

Open Problems

1. Is there a quantum spin system that commutes with the dimer transfer matrix, in some sense?
2. How to get lower bounds?

A partial answer to question 1 is given in

Integrability and conformal data of the dimer model

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The problem is that the "integrals of motion" are not short-ranged quantum spin chains. Also, there is a question about b.c.'s.