Condensation of fermion pairs in a domain

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BCS states

We consider a gas of spin 1/2 fermions, confined to a domain $\Omega \subset \mathbb{R}^d$, at low density and zero temperature. The particles interact via a (somewhat attractive) two body potential.

Assumption: The system state is a BCS (quasi-free) state. It is then fully described by the two operators

$$\gamma = \text{one body density matrix},$$
$$\alpha = \text{pairing wave function}$$

on $L^2(\Omega)$. They satisfy the operator inequalities $0 \leq \gamma \leq 1$ and $\alpha\overline{\alpha} \leq \gamma(1-\gamma)$.

We denote the operator kernels of γ and α by $\gamma(x, y)$ and $\alpha(x, y)$.

BCS energy in a domain

We distinguish two scales, a microscopic one of $O(h)$ and a macroscopic one of $O(1)$.

- **Macroscopic:** Domain Ω; weak external field $h^2 W$.
- **Microscopic:** Kinetic energy of fermions; two body interaction V (attractive enough s.t. $-\Delta + V$ has a bound state).

BCS energy

$$\mathcal{E}_\mu^{\mathrm{BCS}}(\gamma, \alpha) := \mathrm{tr}\left[(-h^2 \Delta_\Omega + h^2 W - \mu)\gamma\right] + \iint_{\Omega^2} V\left(\frac{x-y}{h}\right) |\alpha(x,y)|^2 \mathrm{d}x\mathrm{d}y$$

for "admissible" γ and α. Here $\mu < 0$ is the chemical potential and $-\Delta_\Omega$ is the Dirichlet Laplacian (particles are confined).

Condensate of pairs

Heuristics: μ is chosen s.t. we are at low density. The fermions form tightly bound pairs. Low density $\Rightarrow$ pairs are far apart $\Rightarrow$ pairs look like bosons to one another $\Rightarrow$ pairs form a BEC.

Macroscopic description of BEC is given by Gross-Pitaevskii (GP) energy

$$\mathcal{E}_D^{\mathrm{GP}}(\psi) := \int_\Omega \left(|\nabla\psi|^2 + (W - D)|\psi|^2 + g|\psi|^4 \right) \mathrm{d}x.$$

The minimizer $\psi : \Omega \to \mathbb{R}_+$ is the "order parameter" and describes the macroscopic condensate density.

$D \in \mathbb{R}$ and $g > 0$ are parameters (for us they will be determined by the microscopic BCS theory).

Literature

Goal: Derive the effective, nonlinear GP theory from $\mathcal{E}_\mu^{\mathrm{BCS}}$ as $h \downarrow 0$.

Previous results:

- Hainzl-Seiringer 2012; Hainzl-Schlein 2012; Bräunlich-Hainzl-Seiringer 2016; in this context.
- Frank-Hainzl-Seiringer-Solovej 2012; at positive temperature and density.

Idea of the derivation: Integrate out microscopic relative coordinate $\frac{x-y}{h}$ of fermion pairs. Center-of-mass coordinate $X = \frac{x+y}{2}$ is macroscopic and described by GP theory. (Semiclassical methods.)

The previous results are for systems without boundary, i.e. $\Omega = \mathbb{R}^d$ or Ω is the torus. We are interested in the effect of the Dirichlet boundary conditions on the GP theory.

Main result

Theorem. Assume that the pair binding energy is negative:

$$-E_b := \inf \operatorname{spec}_{L^2(\mathbb{R}^d)}(-\Delta + V) < 0.$$

Set the chemical potential $\mu = -E_b + Dh^2$ for some $D \in \mathbb{R}$. If Ω is nice, then as $h \downarrow 0$,

$$\min_{(\gamma,\alpha) \text{ adm.}} \mathcal{E}^{\mathrm{BCS}}_{-E_b+Dh^2}(\gamma,\alpha) = h^{4-d} \min_{\psi \in H^1_0(\Omega)} \mathcal{E}^{\mathrm{GP}}_D(\psi) + O(h^{4-d+c_\Omega})$$

with c_Ω depending on the regularity of Ω ($c_\Omega > 0$ for bounded Lipschitz domains, $c_\Omega = 1$ for convex domains,...).

Remarks:

- On RHS, minimization over $\psi \in H^1_0(\Omega)$ means the Dirichlet b.c. are preserved for GP energy.
- The choice $\mu = -E_b + Dh^2$ indeed corresponds to low density, by a duality argument.

A linear model problem

A particle pair described by the two body Schrödinger operator

$$H_h := \frac{h^2}{2}\left(-\Delta_{\Omega,x} - \Delta_{\Omega,y}\right) + V\left(\frac{x-y}{h}\right).$$

Goal: Find the g.s. energy of H_h on $L^2(\Omega \times \Omega)$, as $h \downarrow 0$.
Natural to transform H_h into center-of-mass coordinates

$$X := \frac{x+y}{2}, \qquad r := x - y,$$

and use $-\frac{1}{2}\Delta_x - \frac{1}{2}\Delta_y = -\Delta_r - \frac{1}{4}\Delta_X$ to get

$$-h^2\Delta_r + V(r/h) - \frac{h^2}{4}\Delta_X.$$

If H_h were defined on $\mathbb{R}^d$, then the r and X variable would decouple and the g.s. energy would be the sum of those for the r- and X-dependent part.

However, the boundary conditions prevent this decoupling; H_h describes a true two body problem for fixed $h > 0$.

Result for the linear model problem

Good news: X and r decouple again, to the first two orders in h.

Theorem. As $h \downarrow 0$, the two body operator H_h has the g.s. energy

$$\inf \operatorname{spec}_{L^2(\Omega\times\Omega)} H_h = -E_b + D_c h^2 + O(h^{2+\delta}).$$

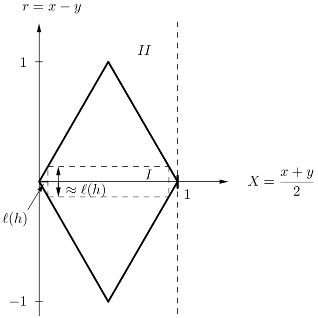
for some $\delta > 0$. Here we defined the g.s. energies in the relative and center-of-mass variables

$$-E_b := \inf \operatorname{spec}_{L^2(\mathbb{R}^d)}(-\Delta + V) < 0,$$

$$D_c := \inf \operatorname{spec}_{L^2(\Omega)} \left(-\frac{1}{4}\Delta_X\right) \in \mathbb{R}.$$

Proof idea for the linear model problem

Let $\Omega = [0,1]$. This becomes a diamond in the (X, r) plane.



Approach to the g.s. energy of H_h: Upper bound from trial state supported in the small rectangle I, where $\ell(h) = h\log(h^{-q}) \gg h$. Uses exponential decay of the Schrödinger eigenfunction $\alpha_0(r/h)$. Lower bound by using that Dirichlet energies go down when domain is increased (to the strip II). Note that X and r decouple on the strip.

Thank you for your attention!