

Zeta Functions of the Dirac Operator on Quantum Graphs

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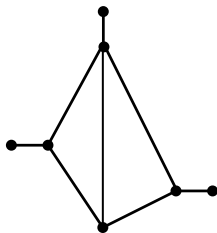
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Metric Graphs

$$\Gamma = \{V, \mathcal{B}, L\}$$



Quantum Graph: Metric Graph + Differential Operator

Dirac Operator

$$\mathcal{D} = -i\alpha \frac{d}{dx_b} + m\beta$$

α and β are 4×4 matrices that satisfy $\alpha^2 = \beta^2 = I$ and $\alpha\beta + \beta\alpha = 0$

Vertex Conditions: $\mathbb{A}\psi^+ + \mathbb{B}\psi^- = \mathbf{0}$

$$\psi^+ = (\psi_1^1(0), \psi_2^1(0), \dots, \psi_2^B(0), \psi_1^1(L_1), \psi_2^1(L_1), \dots, \psi_1^B(L_B), \psi_2^B(L_B))^T$$

$$\psi^- = (-\psi_4^1(0), \psi_3^1(0), \dots, \psi_3^B(0), \psi_4^1(L_1), -\psi_3^1(L_1), \dots, \psi_4^B(L_B), -\psi_3^B(L_B))^T$$

The operator is self-adjoint if and only if \mathbb{A} and \mathbb{B} are $4B \times 4B$ matrices that satisfy

$$\text{rank}(\mathbb{A}, \mathbb{B}) = 4B \text{ and } \mathbb{A}\mathbb{B}^\dagger = \mathbb{B}\mathbb{A}^\dagger.$$

[1] J. Bolte and J. M. Harrison, Spectral statistics for the Dirac operator on graphs, *J. Phys. A: Math. Gen.* **36:2747** (2003).

$$\mathcal{D} = -i\alpha \frac{d}{dx_b} + m\beta$$

$$\alpha = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}$$

Solutions to $\mathcal{D}\psi_k = E(k)\psi_k$ are of the form

$$\psi^b(x_b) = \mu_\alpha^b \begin{pmatrix} 1 \\ 0 \\ 0 \\ i\gamma(k) \end{pmatrix} e^{ikx_b} + \mu_\beta^b \begin{pmatrix} 0 \\ 1 \\ -i\gamma(k) \\ 0 \end{pmatrix} e^{ikx_b} + \hat{\mu}_\alpha^b \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i\gamma(k) \end{pmatrix} e^{-ikx_b} + \hat{\mu}_\beta^b \begin{pmatrix} 0 \\ 1 \\ i\gamma(k) \\ 0 \end{pmatrix} e^{-ikx_b}$$

where

$$\gamma(k) := \frac{E(k) - m}{k} \quad E(k) := \sqrt{k^2 + m^2}$$

Solutions

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where

$$\gamma(k) := \frac{E(k) - m}{k} \quad E(k) := \sqrt{k^2 + m^2}$$

Solutions to $\mathcal{D}\psi_k = -E(k)\psi_k$ are of the form

$$\psi^b(x_b) = \mu_\alpha^b \begin{pmatrix} i\gamma(k) \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{ikx_b} + \mu_\beta^b \begin{pmatrix} 0 \\ -i\gamma(k) \\ 1 \\ 0 \end{pmatrix} e^{ikx_b} + \hat{\mu}_\alpha^b \begin{pmatrix} -i\gamma(k) \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-ikx_b} + \hat{\mu}_\beta^b \begin{pmatrix} 0 \\ i\gamma(k) \\ 1 \\ 0 \end{pmatrix} e^{-ikx_b}$$

Secular Equation

For positive solutions:

$$\det \left(\mathbb{A} + \gamma(k)\mathbb{B} \begin{pmatrix} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{pmatrix} \right) = 0$$

For negative solutions:

$$\det \left(\gamma(k)\mathbb{A} - \mathbb{B} \begin{pmatrix} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{pmatrix} \right) = 0$$

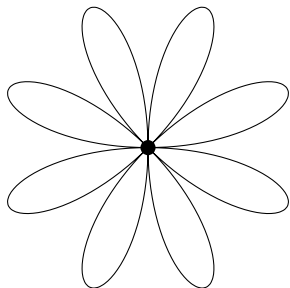
Spectral Zeta Function

Given the set of roots $\{\dots < k_{-2} < k_{-1} < k_1 < k_2 < \dots\}$ of the secular equation, the spectral zeta function is defined as

$$\begin{aligned}\zeta(s) &= 2 \sum'_{j=-\infty}^{\infty} E(k_j)^{-s} \\ &= 2 \sum'_{j=-\infty}^{\infty} k_j^{-s}\end{aligned}$$

in the massless case.

Rose Graph



Vertex Conditions

$$u_o^b \mathbf{v}^b(0) = u_t^b \mathbf{v}^b(L_b) = \eta \quad \text{for all bonds } b$$

$$\sum_{b=1}^B u_o^b \mathbf{w}^b(0) = \sum_{b=1}^B u_t^b \mathbf{w}^b(L_b)$$

where

$$\mathbf{v}^b(x_b) = \begin{pmatrix} \psi_1^b(x_b) \\ \psi_2^b(x_b) \end{pmatrix} \quad \text{and} \quad \mathbf{w}^b(x_b) = \begin{pmatrix} -\psi_4^b(x_b) \\ \psi_3^b(x_b) \end{pmatrix}$$

Secular Equation:

$$\sum_{b=1}^B \frac{\cos \theta_b - \cos kL_b}{\sin kL_b} = 0 \quad \text{where } \cos \theta_b = \frac{1}{2} \text{tr}(u_o^b (u_t^b)^{-1})$$

Spectral Zeta Function

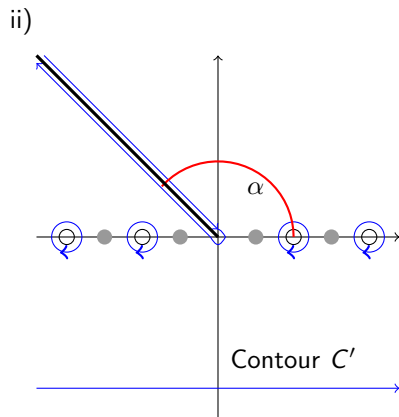
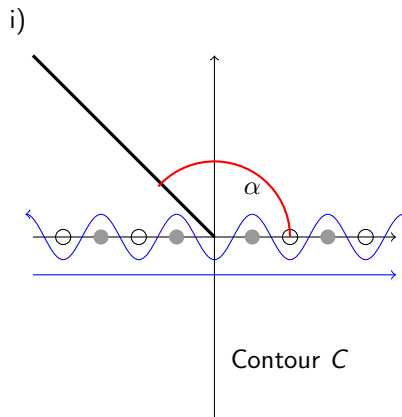
$$f(z) = z \sum_{b=1}^B \frac{\cos \theta_b - \cos zL_b}{\sin zL_b}$$

$$\begin{aligned}\zeta(s) &= 2 \sum'_{j=-\infty}^{\infty} k_j^{-s} \\ &= \frac{1}{i\pi} \int_C z^{-s} \frac{f'(z)}{f(z)} dz \\ &= \frac{1}{i\pi} \int_C z^{-s} \frac{d}{dz} \log f(z) dz\end{aligned}$$

where C is any contour that encloses the zeros of f (while avoiding its poles).

[2] J. Harrison and K. Kirsten, Zeta functions of quantum graphs, *J. Phys. A: Math. Theor.* **44** (2011).

Contours



The shaded circles are the zeros of f
and the empty circles are the poles.

Spectral Zeta Function – Rose Graph

$$\zeta(s) = \zeta_p(s) + \zeta_I(s) + \zeta_b(s)$$

$$\begin{aligned}\zeta_p(s) &= 2 \sum_{b=1}^B \left(\sum_{n=-\infty}^{-1} \left(\frac{n\pi}{L_b} \right)^{-s} + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L_b} \right)^{-s} \right) \\ &= 2(e^{-i\pi s} + 1)\zeta_R(s) \sum_{b=1}^B \left(\frac{\pi}{L_b} \right)^{-s}\end{aligned}$$

$$\zeta_I(s) = 0 \quad \text{if } \operatorname{Re}(s) > 0$$

Spectral Zeta Function – Rose Graph

$$\zeta_b(s) = e^{i(\pi-\alpha)s} \frac{2 \sin \pi s}{\pi} \int_0^\infty u^{-s} \frac{d}{du} \log f(ue^{i\alpha}) du$$

This converges for $0 < \operatorname{Re}(s) < 2$.

$$\zeta_b(s) = e^{i(\pi-\alpha)s} \frac{2 \sin \pi s}{\pi} \int_0^\infty u^{-s} \frac{d}{du} \log \left(ue^{i\alpha} \hat{f}(u) \right) du$$

$$\hat{f}(u) = \sum_{b=1}^B \frac{\cos \theta_b - \cos L_b e^{i\alpha} u}{\sin L_b e^{i\alpha} u}$$

Spectral Zeta Function – Rose Graph

Theorem

$$\zeta(s) = e^{i(\pi-\alpha)s} \frac{2 \sin s\pi}{\pi} \left[\int_0^1 u^{-s} \frac{d}{du} \log \left(u e^{i\alpha} \hat{f}(u) \right) du + \frac{1}{s} \right. \\ \left. + \int_1^\infty u^{-s} \frac{d}{du} \log \hat{f}(u) du \right] + 2(e^{-i\pi s} + 1) \zeta_R(s) \sum_{b=1}^B \left(\frac{\pi}{L_b} \right)^{-s}$$

where $\text{Re}(s) < 2$ and

$$\hat{f}(u) = \sum_{b=1}^B \frac{\cos \theta_b - \cos L_b e^{i\alpha} u}{\sin L_b e^{i\alpha} u}.$$

Spectral Determinant – Rose Graph

$$\begin{aligned}\det'(\mathcal{D}) &= \prod_{j=-\infty}^{\infty}{}' k_j^2 \\ &= \exp(-\zeta'(0)) \\ &= \frac{(2\pi)^{2B}(-1)^{B+1}}{B^2} \left(\sum_{b=1}^B \frac{\cos \theta_b - 1}{L_b} \right)^2 \prod_{b=1}^B \left(\frac{L_b}{\pi} \right)^2\end{aligned}$$

Spectral Zeta Function – General Graph Without Mass

Theorem

$$\zeta(s) = e^{i(\pi-\alpha)s} \frac{2 \sin s\pi}{\pi} \left[\int_0^1 u^{-s} \frac{d}{du} \log \left((ue^{i\alpha})^{4B-1} \hat{f}(u) \right) du + \frac{4B-1}{s} \right. \\ \left. + \int_1^\infty u^{-s} \frac{d}{du} \log \hat{f}(u) du \right] + 2(e^{-i\pi s} + 1) \zeta_R(s) \sum_{b=1}^B \left(\frac{\pi}{L_b} \right)^{-s}$$

where $\operatorname{Re}(s) < M$ and

$$\hat{f}(u) = \det \left(\mathbb{A} + \mathbb{B} \begin{pmatrix} \cot ue^{i\alpha} L & -\csc ue^{i\alpha} L \\ -\csc ue^{i\alpha} L & \cot ue^{i\alpha} L \end{pmatrix} \right).$$

Spectral Determinant – General Graph Without Mass

$$\det'(\mathcal{D}) = \frac{c_0^2(-1)^B}{\det(\mathbb{A} - i\mathbb{B})^2} \prod_{b=1}^B (2L_b)^2$$

$$c_0 = f(0) \neq 0$$

Spectral Zeta Function – General Graph With Mass

Given the set of roots $\{k_1, k_2, \dots\}$ of the positive energy secular equation and the set of roots $\{\tilde{k}_1, \tilde{k}_2, \dots\}$ to the negative energy secular equation, the spectral zeta function is defined as

$$\begin{aligned}\zeta(s) &= 2 \sum_{j=1}^{\infty}{}' E(k_j)^{-s} + 2 \sum_{j=1}^{\infty}{}' \left(-E(\tilde{k}_j)\right)^{-s} \\ &= 2 \sum_{j=1}^{\infty} \left(\sqrt{k_j^2 + m^2}\right)^{-s} + 2 \sum_{j=1}^{\infty} \left(-\sqrt{\tilde{k}_j^2 + m^2}\right)^{-s} \\ &= \zeta^+(s) + \zeta^-(s).\end{aligned}$$

General Graph With Mass

Positive Eigenvalues:

$$f(z) = \det \left(\mathbb{A} + \gamma(z)\mathbb{B} \begin{pmatrix} \cot zL & -\csc zL \\ -\csc zL & \cot zL \end{pmatrix} \right)$$

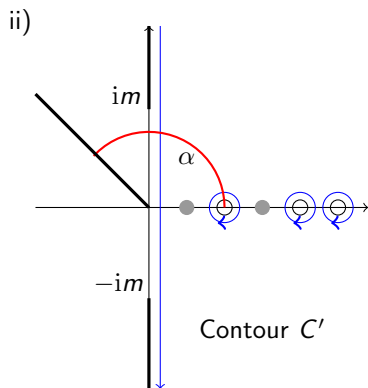
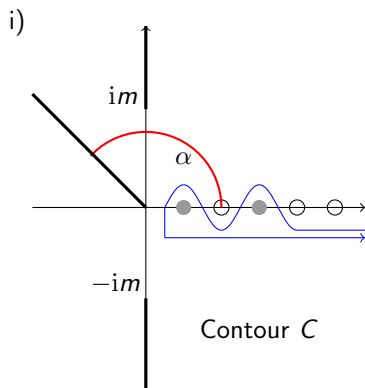
$$\zeta^+(s) = \frac{1}{i\pi} \int_C (z^2 + m^2)^{-s/2} \frac{d}{dz} \log f(z) dz$$

Negative Eigenvalues:

$$g(z) = \det \left(\gamma(z)\mathbb{A} - \mathbb{B} \begin{pmatrix} \cot zL & -\csc zL \\ -\csc zL & \cot zL \end{pmatrix} \right)$$

$$\zeta^-(s) = \frac{(-1)^{-s}}{i\pi} \int_C (z^2 + m^2)^{-s/2} \frac{d}{dz} \log g(z) dz$$

Contour



The shaded circles are the zeros of f/g and the empty circles are the poles.

Spectral Zeta Function – General Graph With Mass

$$\begin{aligned}\zeta_p^+(s) &= 2 \sum_{b=1}^B \sum_{n=1}^{\infty} \left(\left(\frac{n\pi}{L_b} \right)^2 + m^2 \right)^{-s/2} \\ &= 2 \sum_{b=1}^B \left(\frac{\pi}{L_b} \right)^{-s} E \left(\frac{s}{2}, \left(\frac{mL_b}{\pi} \right)^2 \right)\end{aligned}$$

$$\zeta_b^+(s) = \frac{2}{\pi} \sin \left(\frac{\pi s}{2} \right) \int_m^{\infty} (t^2 - m^2)^{-s/2} \frac{d}{dt} f(it) dt$$

which converges for $-1 < \operatorname{Re}(s) < 1$.

Spectral Zeta Function – General Graph With Mass

Theorem

$$\begin{aligned}\zeta(s) &= 2(1 + (-1)^{-s}) \sum_{b=1}^B \left(\frac{\pi}{L_b}\right)^{-s} E\left(\frac{s}{2}, \left(\frac{mL_b}{\pi}\right)^2\right) \\ &+ \frac{2}{\pi} \sin\left(\frac{\pi s}{2}\right) \left[\int_m^\infty (t^2 - m^2)^{-s/2} \frac{d}{dt} \log \hat{f}(t) dt \right. \\ &\left. + (-1)^{-s} \int_m^\infty (t^2 - m^2)^{-s/2} \frac{d}{dt} \log \hat{g}(t) dt \right]\end{aligned}$$

where $-1 < \operatorname{Re}(s) < 1$ and

$$\begin{aligned}\hat{f}(t) &= \det \left(\mathbb{A} + \hat{\gamma}(t) \mathbb{B} \begin{pmatrix} \coth tL & -\operatorname{csch} tL \\ -\operatorname{csch} tL & \coth tL \end{pmatrix} \right) \\ \hat{g}(t) &= \det \left(\hat{\gamma}(t) \mathbb{A} - \mathbb{B} \begin{pmatrix} \coth tL & -\operatorname{csch} tL \\ -\operatorname{csch} tL & \coth tL \end{pmatrix} \right), \text{ and} \\ \hat{\gamma}(t) &= \frac{\sqrt{t^2 - m^2} + im}{t}.\end{aligned}$$

Summary

- We found a formulation of the spectral zeta function of the Dirac operator using a contour integral technique.
- In the case of zero mass, we analytically continued our expression to a domain including $s = 0$ and calculated the zeta-regularized spectral determinant.
- We did this first for a rose graph without mass, and then for a general graph with and without mass.

- [1] J. Bolte and J. M. Harrison, Spectral statistics for the Dirac operator on graphs, *J. Phys. A: Math. Gen.* 36(11):[2747-2769](#) (2003).
- [2] J. Harrison and K. Kirsten, Zeta functions of quantum graphs, *J. Phys. A: Math. Theor.* 44(33):[235301](#), [29](#) (2011).
- [3] J. Harrison, T. Weyand, and K. Kirsten, Zeta functions of the Dirac Operator on quantum graphs, *J. Math. Phys.* 57(10):[102301](#), [17](#) (2016).

