Non-self-adjoint graphs

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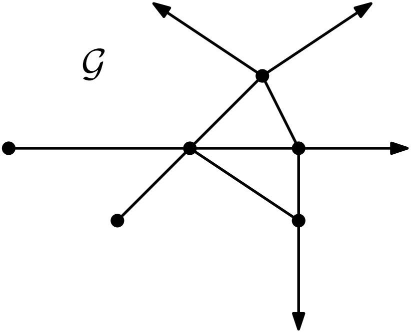
Based on

[1] A. Hussein, D. Krejčiřík and P. Siegl: *Non-self-adjoint graphs*, Transactions of the AMS 367, (2015) 2921-2957

Outline

1. Introduction and "the example"
2. Classes of boundary conditions
3. Spectral properties
4. Similarity transforms

$$-\Delta \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \quad + \quad \text{boundary/vertex conditions} \quad \text{in} \quad L^2(\mathcal{G})$$



$$L^2(\mathcal{G}) = \oplus_{j=1}^{N} L^2((0, a_j)), \quad a_j \in (0, +\infty], \quad N = \#\text{edges} < \infty$$

Non-self-adjoint graphs

- "fundamental non-self-adjointness":
  - non-symmetric boundary/vertex conditions, e.g. complex δ-interactions
  - no problems with too little or too many conditions

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[2] A. Hussein. *J. Evol. Equ.* 14 (2014), pp. 477–497; V. Kostrykin, J. Potthoff, and R. Schrader. *J. Math. Phys.* 53 (2012), p. 095206; V. Kostrykin, J. Potthoff, and R. Schrader. *Proc. Symp. in Pure Math.* 77 (2008), pp. 423–458; V. Kostrykin, J. Potthoff, and R. Schrader. *Adventures in math. phys.* Vol. 447. Contemp. Math. AMS, Providence, 2007, pp. 175–198.

[3] P. Freitas and J. Lipovský. arXiv:1307.6377.

Non-self-adjoint graphs

- "fundamental non-self-adjointness":
  - non-symmetric boundary/vertex conditions, e.g. complex δ-interactions
  - no problems with too little or too many conditions
- motivation for complex potentials/interactions:
  - electromagnetism, optics with losses and gains
  - superconductivity, damped wave equation
  - stochastic processes
  - open quantum systems, effective models
- existing literature
  - non-self-adjoint point interactions[1]
  - m-accretive and m-dissipative graphs[2]
  - damped wave equation on graphs[3]

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Minimal and maximal operators

- minimal operator

$$\begin{aligned} \mathrm{Dom}(-\Delta_{\min}) &= W_0^{2,2}(\mathcal{G}) := \oplus_{j=1}^{N} W_0^{2,2}((0,a_j)) \\ (-\Delta_{\min}\psi)_j &:= -\psi_j'' \end{aligned}$$

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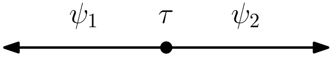
Our Laplacians

$$-\Delta_{\min} \subset -\Delta_{\mathcal{M}} \subset -\Delta_{\max}, \quad \Delta_{\mathcal{M}} \neq \Delta_{\mathcal{M}}^*$$
$$\mathrm{Dom}(-\Delta_{\mathcal{M}}) := \{\psi \in \mathrm{Dom}(-\Delta_{\max}) : [\psi] \oplus [\psi'] \in \mathcal{M} \subset \mathbb{C}^{2d}\}$$
$$\text{we assume :} \quad \dim \mathcal{M} = d$$

“The example”

τ-interaction[4]



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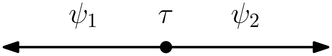

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- $\tau = 0$:
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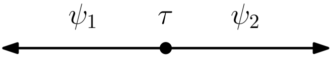

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    - $Q_{\mathcal{M}}[\psi] = \|\psi'\|^2_{L^2(\mathcal{G})} + (1 - e^{2\mathrm{i}\tau})\psi_2(0)\,\boxed{\overline{\psi_2'(0)}}$
    - cannot be defined through sectorial forms: $\boxed{\mathrm{Num}(-\Delta_{\mathcal{M}}) = \mathbb{C}}$

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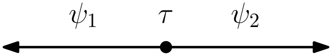

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- $\tau = \pi/2$:
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Boundary conditions

- subspaces $\mathcal{M}$ parametrized by matrices $A, B \in \mathbb{C}^{d\times d}$, $\mathcal{M} = \mathcal{M}(A, B)$

$$\mathrm{Dom}(-\Delta(A,B)) = \left\{\psi \in \mathrm{Dom}(-\Delta_{\max}) : A[\psi] + B[\psi'] = 0\right\}$$

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$\Longleftrightarrow$ (A,B) parametrization[5]: $AB^* = BA^*$

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$\Longleftrightarrow$ m-sectorial parametrization[6]: $(A, B) \simeq (L + P, P^{\perp})$

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Regular boundary conditions

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m-sectorial boundary conditions

- regular BC are called m-sectorial if $(A, B) \simeq (L + P, P^{\perp})$
- all self-adjoint BC are m-sectorial
- τ-interaction is m-sectorial iff $\tau = 0$

Totally degenerated[7] BC - irregular

- 
- BC: $\psi(0) = 0$, $\psi'(0) = 0$, $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
- $\dim \mathcal{M}(A, B) = 2 = d$ and $A + \mathrm{i}kB$ is not invertible for any $k \in \mathbb{C}$
- spectral pathology: $\boxed{\sigma(-\Delta(A, B)) = \emptyset}$

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Examples

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Indefinite Laplacian[8] - irregular

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- BC: $\psi_1(0) = \psi_2(0), \psi_1'(0) = \psi_2'(0)$, $\quad A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$
- spectral pathology: $\boxed{\sigma(-\Delta(A, B)) = \mathbb{C}}$
- $-\Delta(A, B) \simeq -\operatorname{sgn}(x) \frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sgn}(x) \frac{\mathrm{d}}{\mathrm{d}x}$ in $L^2(\mathbb{R})$

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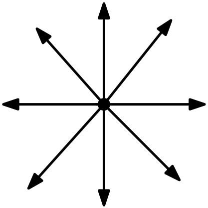
Complex δ-interaction - m-sectorial



$$\psi_1(0) = \psi_2(0) = \cdots = \psi_N(0)$$

$$\sum_{i=1}^{N} \psi_i'(0) = \gamma \psi_1(0), \quad \gamma \in \mathbb{C}$$

$$A = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -\gamma & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

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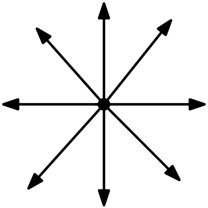

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$$\sigma(-\Delta(A,B)) = \begin{cases} \{-(\gamma/N)^2\} \cup [0,\infty), & \text{if } \operatorname{Re}\gamma < 0 \\ [0,\infty), & \text{if } \operatorname{Re}\gamma \geq 0 \end{cases}$$

Point spectrum

- $\overline{\sigma_{\mathrm{p}}(-\Delta(A,B))}$ is discrete set ($\neq \mathbb{C}$) and

$$\lambda \in \sigma_{\mathrm{p}}(-\Delta(A,B)) \setminus [0,\infty) \Longleftrightarrow \overline{\lambda} \in \sigma_{\mathrm{p}}(-\Delta(A,B)^*) \setminus [0,\infty)$$

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Residual spectrum

- $\sigma_{\mathrm{r}}(-\Delta(A,B)) \subset [0,\infty)$ (discrete subset)
- $\sigma_{\mathrm{r}}(-\Delta(A,B)) = \emptyset$ if there are no bounded/unbounded edges
- $\sigma_{\mathrm{r}}(-\Delta(A,B))$ may be non-empty!

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Essential spectrum

- warning: at least 5 different definitions of essential spectrum of nsa operators!
- σ_{e5}: complement of isolated eigenvalues of finite algebraic multiplicity

$$\sigma_{\mathrm{e5}}(-\Delta(A,B)) = \begin{cases} \emptyset & \text{if there are no unbounded edges} \\ [0,\infty) & \text{if there is an unbounded edge} \end{cases}$$

M-sectorial complex Robin BC[9]

$$\psi'(0) + \mathrm{i}\alpha\psi(0) = 0 \ \psi'(\pi) + \mathrm{i}\alpha\psi(\pi) = 0, \qquad \alpha \in \mathbb{R}$$

$$A = \begin{bmatrix} \mathrm{i}\alpha & 0 \\ 0 & -\mathrm{i}\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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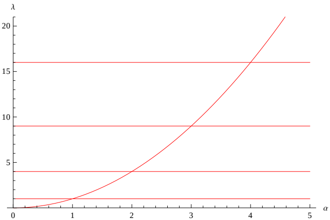

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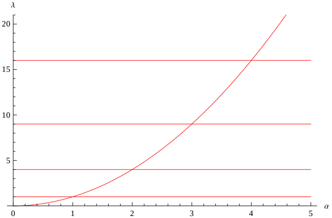

More than real spectrum

- eigenfunctions $\{\psi_n\}_{n\in\mathbb{N}_0}$ form a Riesz basis

$\Rightarrow$ similarity to a self-adjoint operator

$$-\Delta(A,B) \sim -\Delta_{\mathrm{N}} + \frac{\alpha^2}{\pi}\langle\cdot,1\rangle, \quad \alpha \notin \mathbb{Z}\setminus\{0\}$$

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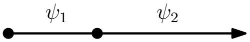
Graph with residual spectrum



$$-\Delta(A,B)$$

$$\psi_1'(0) + \mathrm{i}\alpha\psi_1(0) = 0$$

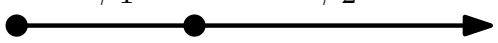
$$\psi_1'(\pi) + \mathrm{i}\alpha\psi_1(\pi) = 0$$

$$-\psi_1(\pi) = \psi_2'(0)$$

$$A = \begin{bmatrix} \mathrm{i}\alpha & 0 & 0 \\ 0 & -\mathrm{i}\alpha & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spectral properties

Graph with residual spectrum



$$-\Delta(A,B) \qquad\qquad -\Delta(A,B)^*$$

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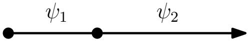

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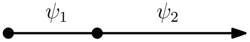

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- same spectra: $\sigma(-\Delta(A,B)) = \sigma(-\Delta(A,B)^*) = [0,\infty)$
- but for point spectra:

$$\sigma_{\mathrm{p}}(-\Delta(A,B)) = \emptyset \qquad \text{vs.} \qquad \sigma_{\mathrm{p}}(-\Delta(A,B)^*) = \{\alpha^2\} \cup \{n^2\}_{n\in\mathbb{N}}$$

$\Rightarrow$ $\boxed{\sigma_{\mathrm{r}}(-\Delta(A,B)) = \{\alpha^2\} \cup \{n^2\}_{n\in\mathbb{N}}}$

Recall:

$$\sigma_{\mathrm{r}}(-\Delta(A,B)) = \{\lambda \notin \sigma_{\mathrm{p}}(-\Delta(A,B)) : \overline{\lambda} \in \sigma_{\mathrm{p}}(-\Delta(A,B)^*)\}$$

Compact m-sectorial graphs

- discrete spectrum & Riesz basis of finite dimensional invariant subspaces
- in very special cases (complex Robin BC) $\Rightarrow$ similarity to normal (or self-adjoint) operator but typically not a graph Laplacian

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Similarity of graph Laplacians: assumptions

i) restriction on graphs: all bounded edges of the same length

ii) similarity of matrices A, B and A', B'

$$A' = G^{-1}AG, \quad B' = G^{-1}BG$$

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Theorem: similarity of graph Laplacians

$$\boxed{-\Delta(A', B') = \Phi_G^{-1}(-\Delta(A', B'))\Phi_G}$$

where

$$(\Phi_G\psi)(x_j) := \sum_{i=1}^{N} G_{ji}\psi_i(x_j)$$

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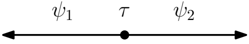

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Similarity transforms

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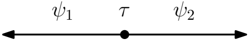

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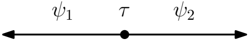

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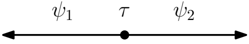

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Regular vs. irregular boundary/vertex conditions

- "usual" spectrum vs. possible pathologies $\sigma(-\Delta) = \emptyset/\mathbb{C}$
- possibly (discrete) non-empty residual spectrum in $[0, \infty)$
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[10] B. Mityagin and P. Siegl. arxiv:1608.00224. 2016.

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Outlook

??? Schrödinger operators on graphs: $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ on edges

??? pseudospectral analysis

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