Georgia Tech Mathematics College of Sciences

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QMath13: Mathematical Results in Quantum Physics

Pointwise control of eigenfunctions on quantum graphs

Evans Harrell
Georgia Tech
www.math.gatech.edu/~harrell

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Abstract

Pointwise bounds on eigenfunctions are useful for establishing localization of quantum states, and they have implications for the distribution of eigenvalues and for physical properties such as conductivity. In the low-energy regime, localization is associated with exponential decrease through potential barriers. We adapt the Agmon method to control this tunneling effect for quantum graphs with Sobolev and pointwise estimates. It turns out that as a generic matter, the rate of decay is controlled by an Agmon metric related to the classical Liouville-Geen approximation for the line, but more rapid decay is typical, arising from the geometry of the graph. In the high-energy regime one expects states to oscillate but to be dominated by a 'landscape function' in terms of the potential and features of the graph. We discuss the construction of useful landscape functions for quantum graphs.

This is joint work with Anna Maltsev of the University of Bristol, *CMP*, to appear, and work in progress

Why do eigenfunctions localize?

1. The tunneling effect.
2. Randomness (Anderson localization).

Quantum graphs

- Vertices are connected by edges, on which $-\psi'' + V(x)\psi = E\psi$.
- The solutions are continuous and connected at the vertices by conditions such as

$$\sum_{e \sim v} f_e'(v^+) = 0.$$

"Kirchhoff conditions"

Quantum graphs

- Kirchhoff conditions correspond to the energy form

$$\phi \to \sum_e \int_e \left(|\phi'|^2 + V(x)|\phi|^2\right) dx$$

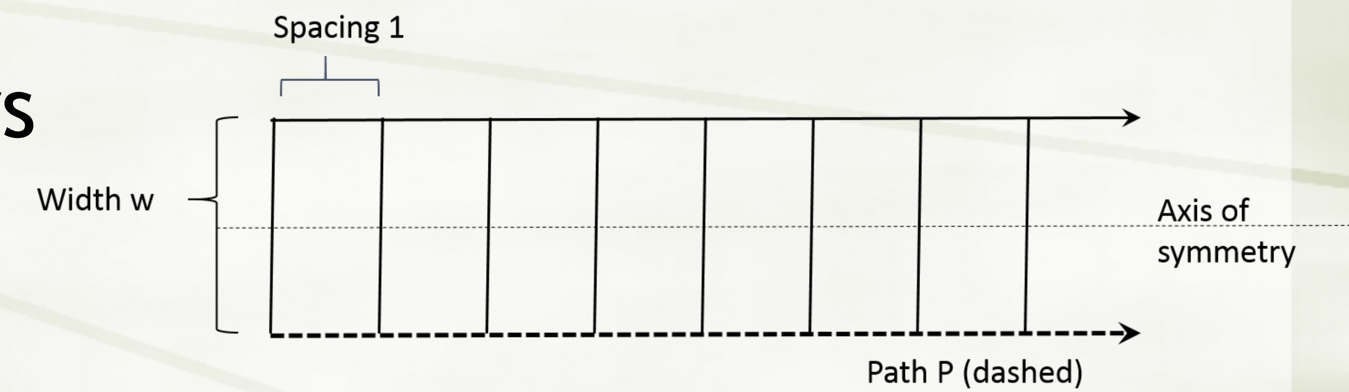
on $H^1(\Gamma)$. (Like Neumann BC)

Infinite quantum graphs

- If the graph tends to ∞, and $E < \liminf V$, how well localized are the eigenfunctions?
- What changes are needed in the Agmon theory due to the connectedness?

Examples

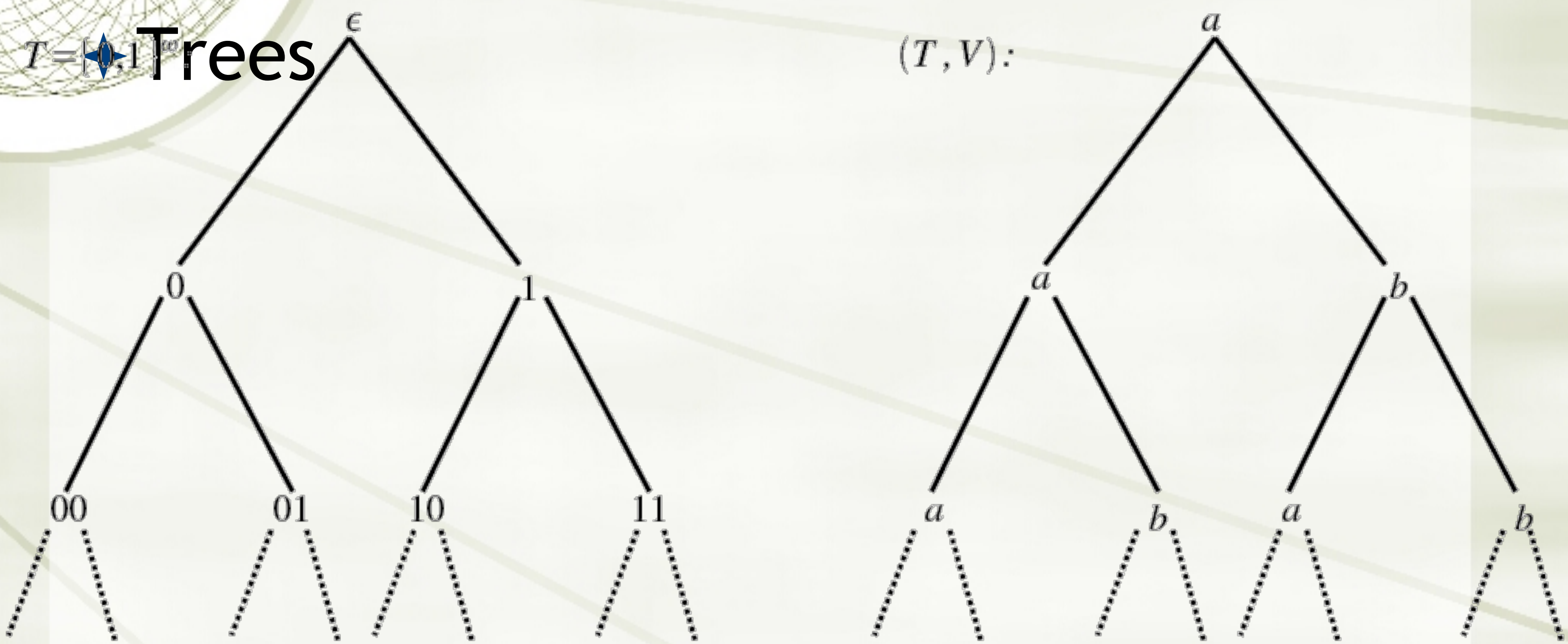
- Ladders



- Let V=0, E=-1 (outside a finite region). There is a symmetric solution that looks like e^{-x} on the sides and constant on the rungs, and an antisymmetric one of the form $ge^{-|\ln \lambda_-|x}$ where g is periodic and $|\ln \lambda_-| > 1$.

Examples

- Trees



Examples

- Trees
  - With branching number b and length L, the transfer matrix for the regular tree has smaller eigenvalue

$$= \frac{1}{\left(\frac{b}{2}+\frac{1}{2}\right)\cosh kL + \sqrt{\left(\left(\frac{b}{2}+\frac{1}{2}\right)\cosh kL\right)^2 - b}}$$

$$< \frac{1}{b\cosh kL}$$

(Here, $E = k^2$.)

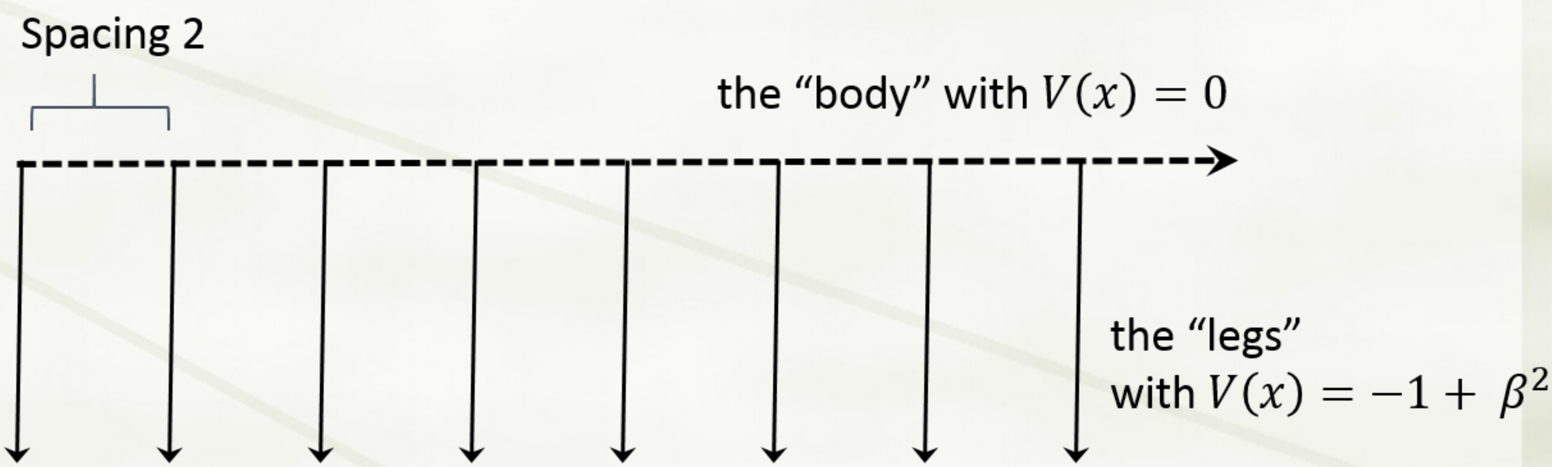
Examples

- Trees
  - We also work out the example of a 2-lengths regular tree.

(Here, $E = k^2$.)

Examples

- Millipedes

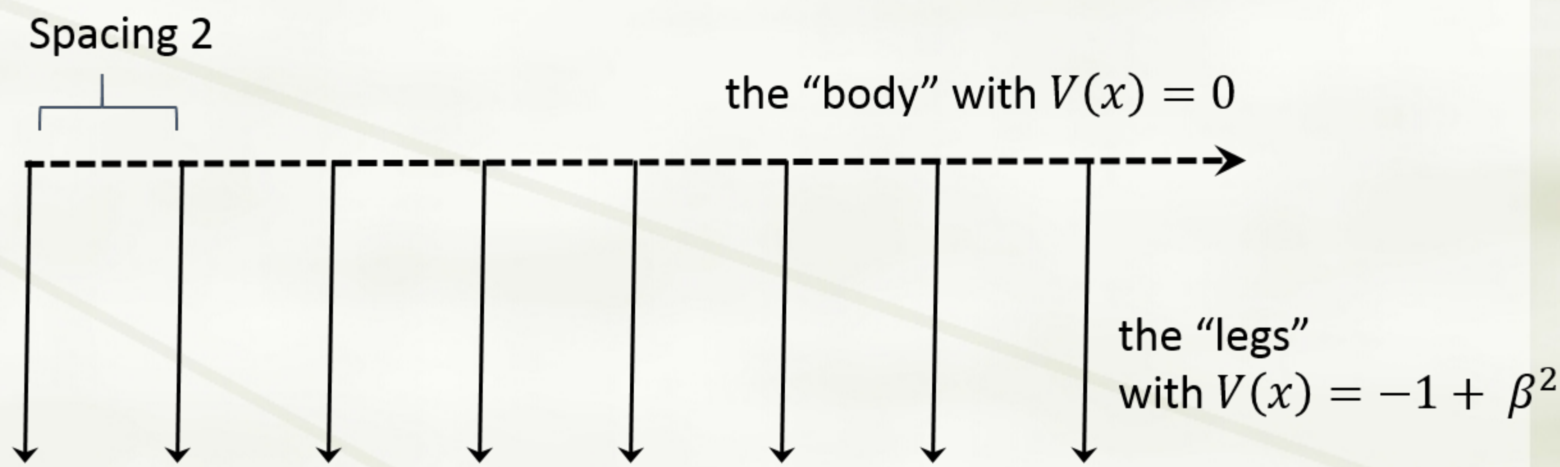


$$\lambda_- = e^{-2}\left(1 - \frac{\beta}{2}\right) + 0(\beta^2)$$

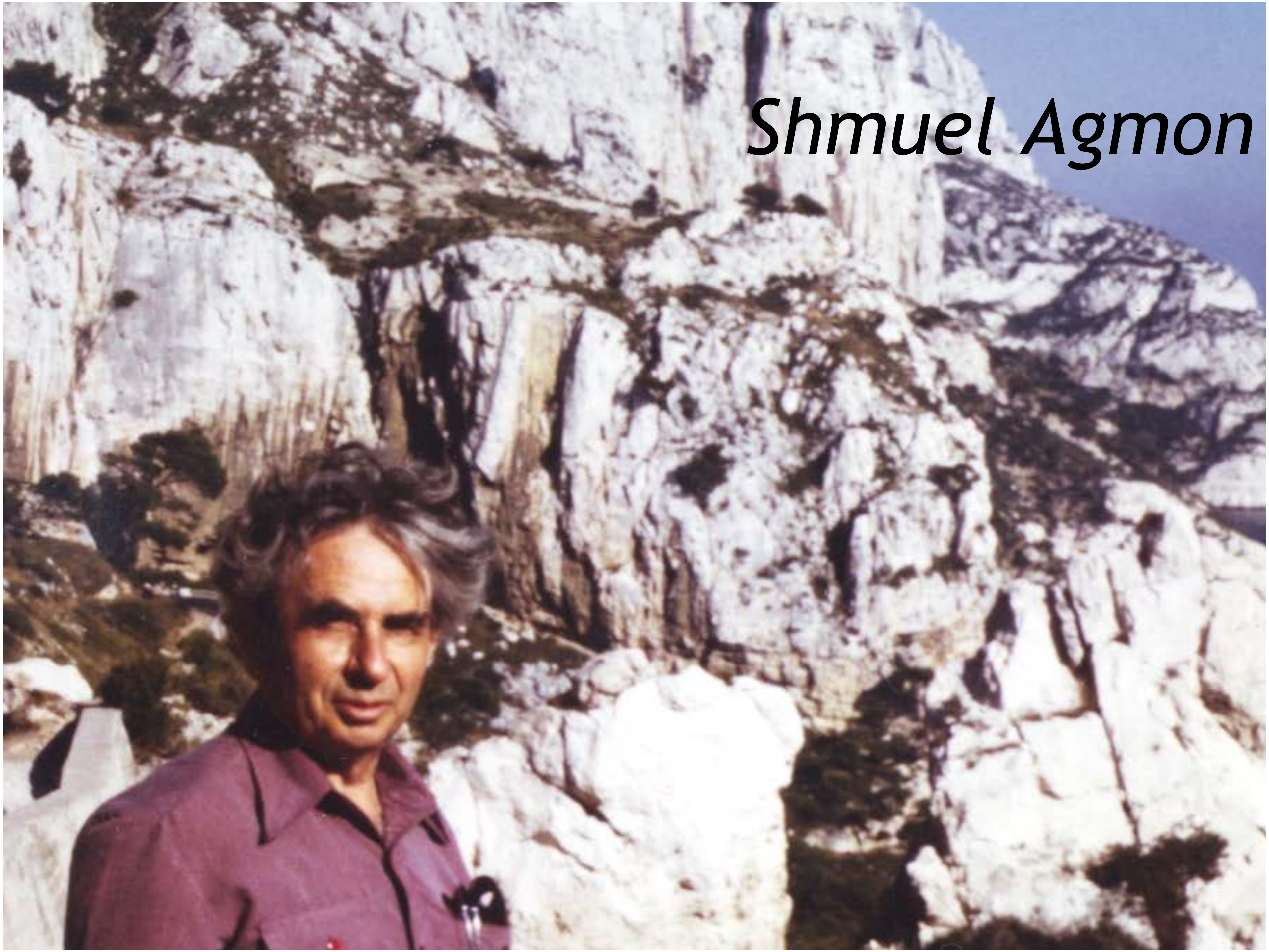
$$= e^{-2-\frac{\beta}{2}+0(\beta^2)}.$$

Examples

- Millipedes



The spectral problem is equivalent to a problem on a half line, with delta potentials at regular intervals.


Shmuel Agmon

The Agmon philosophy

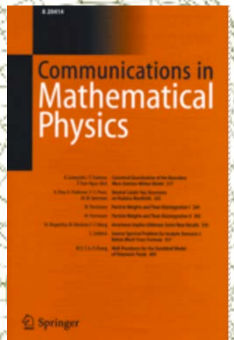
- Exponential decay of eigenfunctions is a geometric concept.
- In the 80’s, Agmon produced many-dimensional estimates that resemble Liouville-Green in 1D.
- Exponential dichotomy – in the nonoscillatory regime one expects asymptotic behavior like

  $\exp(\pm\rho(x))$.
- The Agmon metric depends on the potential ***and, as we shall show, the graph structure***.

The Agmon philosophy

- Basically, if $\psi \in L^2$ and the EVE is valid, you look for a function $F>0$ for which integration by parts identities imply $F\psi \in L^2$. In the classic case $F = e^{\rho}$, where

$$\rho_a(y, x; E) := \min_{\text{paths } P \; y \text{ to } x} \int_P (V(t) - E)_+^{1/2} dt.$$



Our results

1. The "classical action" estimate for eigensolutions on the line is valid for graphs.

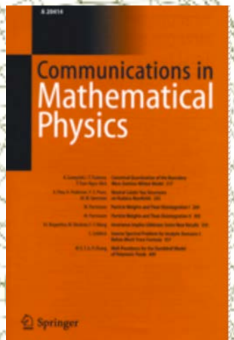
Theorem 1.1. *Suppose that* $\Gamma_0 \subset \Gamma$ *is a connected, infinite subgraph on which* $\liminf(V(x) - E) > 0$. *If* $\psi \in L^2(\Gamma) \cap \mathcal{K}(\Gamma_0)$ *satisfies*

$$-\psi'' + V(x)\psi = E\psi$$

on the edges of Γ_0, *then for any* $\delta < \liminf(V - E)$,

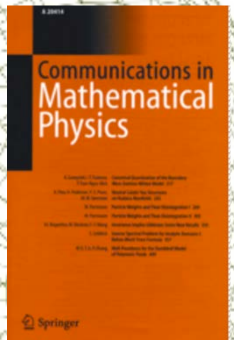
$$e^{\rho_a(x;E-\delta)}\psi \in H^1(\Gamma_0) \cap L^\infty(\Gamma_0). \tag{9}$$

$$\rho_a(y, x; E) := \min_{\text{paths } P\, y \text{ to } x} \int_P (V(t) - E)_+^{1/2} dt.$$



Our results

1. The "classical action" estimate for eigensolutions on the line is valid for graphs.
2. Along a path, a refined estimate is possible in terms of the "fractions of the derivative" p_k. (Here we need assumptions that imply that eigenfunctions decay without changing sign.)



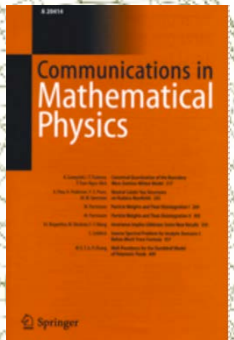
Our results

Theorem 1.2. *Suppose that $\Gamma_0 \subset \Gamma$ is a connected, infinite subgraph on which $\liminf(V(x) - E) > 0$ and that $\psi \in L^2(\Gamma) \cap \mathcal{K}(\Gamma_0)$ satisfies*

$$-\psi'' + V(x)\psi = E\psi$$

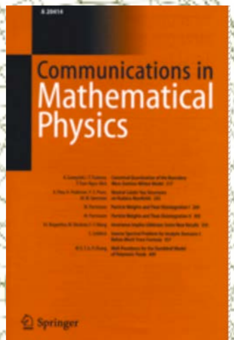
on the edges of Γ_0 and $\psi' < 0$ outside of a set of compact support. Consider any infinite path $P \subset \Gamma_0$, on which the fraction of the derivative exiting from a vertex v is designated p_v. Then for any $\delta < \liminf(V - E)$, $e^{\rho_P(x,E-\delta)}\psi \in L^2(P)$. That is,

$$\sqrt{\prod_{v \in P} \frac{1}{p_v}} e^{\rho_a(x,E-\delta)}\psi \in L^2(P) \cap L^\infty(P).$$



Our results

1. The “classical action” estimate for eigensolutions on the line is valid for graphs.
2. Along a path, a refined estimate is possible.
3. On sufficiently regular graphs, an averaged wave function must decay more rapidly than the classical-action estimate.



Our results

Theorem 1.3. *Suppose that Ψ is the averaged eigenfunction on a quantum graph with regular topology corresponding to a solution ψ of (1), for which $\psi \in L^2(\Gamma) \cap \mathcal{K}$, and that for all x such that $\operatorname{dist}(0, x) = y$, $V(x) \geq V_m(y)$, where $\liminf(V_m(y) - E) > 0$. Define*

$$F_{\text{ave}}(y, E) := \left(\prod_{j : v_j < y} \sqrt{\frac{b_j}{a_j}} \right) e^{\int_0^y \sqrt{V_m(t) - E}\, dt}. \tag{12}$$

Then for each $0 < \delta < \liminf(V_m - E)$,

$$F_{\text{ave}}(y, E - \delta)\Psi \in H^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+).$$

An Agmon identity

- A cleaned-up version of the identities of Harrell-Maltsev proceeds by calculating in two ways the quantity:

$$F^2(x)\eta(x)\psi(x)\left(H - E\right)\eta(x)\psi(x)$$

An Agmon identity

$$\psi^2(x)\left(\eta'(x)\left(\eta(x)F^2(x)\right)'\right) - G'(x) = \left((F\eta\psi)'\right)^2 + \left(V - E - \left(\frac{F'}{F}\right)^2\right)(F\eta\psi)^2 - H'(x)$$

For Agmon estimates F is taken as something like exp(S) for an action integral S, η is a smooth cut-off, and, if we take care about the vertex conditions, the other items will integrate to 0.

An Agmon identity

$$\psi^2(x)\left(\eta'(x)\left(\eta(x)F^2(x)\right)'\right) - G'(x) = \left((F\eta\psi)'\right)^2 + \left(V - E - \left(\frac{F'}{F}\right)^2\right)(F\eta\psi)^2 - H'(x)$$

In the tunneling regime V > E, the integral of the right is comparable to the square of the Sobolev norm of $F\psi$, while the left side will depend only on values in supp(η'), some small interval we can choose.

L^2 and L^∞ estimates

$$
\begin{aligned}
|\phi(x_0)| = |\chi(x_0)\phi(x_0)| &= \left|\int_{x_0-\frac{\ell_{\min}}{2}}^{x_0} (\chi(y)\phi(y))' dy\right| \\
&= \left|\int_{x_0-\frac{\ell_{\min}}{2}}^{x_0} \left(\chi'(y)\phi(y)+\chi(y)\phi'(y)\right) dy\right| \\
&\leq \frac{1}{2}\int_{x_0-\frac{\ell_{\min}}{2}}^{x_0} \left((\chi')^2(y)+(\phi(y))^2+(\chi(y))^2+(\phi'(y))^2\right) dy,
\end{aligned}
$$

Comparison with examples

1. The ladder shows that the classical-action bound is sometimes best possible.
2. The millipede has decay faster than the classical-action bound, and our path-dependent estimate captures that.
3. The regular tree shows that the averaged bound is sharp. (Even one with two lengths.)

"Landscape functions" and E > V.

- Work in the last few years by Filoche and Mayboroda; more recently by Steinerberger.
- A simple or easily calculated function such that $|\psi(x)| \leq L(x)$, and the shape of L(x) closely controls the eigenfunctions.
- They mainly treat finite domain problems in ≥2D.

"Landscape functions" and E > V.

- The most typical Landscape functions are solutions of H L(x) = 1, H = -Δ + V(x), V(x) ≥ 0, for then if

$$W(x) := \pm\psi(x) - E\|\psi\|_\infty L(x),$$

$$HW(x) = E\left(\pm\psi(x) - \|\psi\|_\infty\right) \leq 0,$$

so by the maximum principle, W(x) ≤ 0.

“Landscape functions” and E > V.

- W(x) ≤ 0 means that

$$|\psi(x)| \leq E\|\psi\|_\infty L(x).$$

- To adapt this to quantum graphs seems at first hopeless, because QGs are locally one-dimensional, and this estimate is useless in 1D, since if L>0 and $L'' = V L - 1 < 0$, L cannot wiggle.

"Landscape functions" and E > V.

- With Maltsev, we are piecing together landscape functions.
  - In the tunneling region where $V - E \geq \delta > 0$, we get bounds of the form
  - In transition regions we can control the growth of an eigenfunction by local, quantitative Harnack estimates. (Bounds on max over min of function)
  - Near bottoms of wells we use the maximum principle trick, but only locally.

Local control in 1D

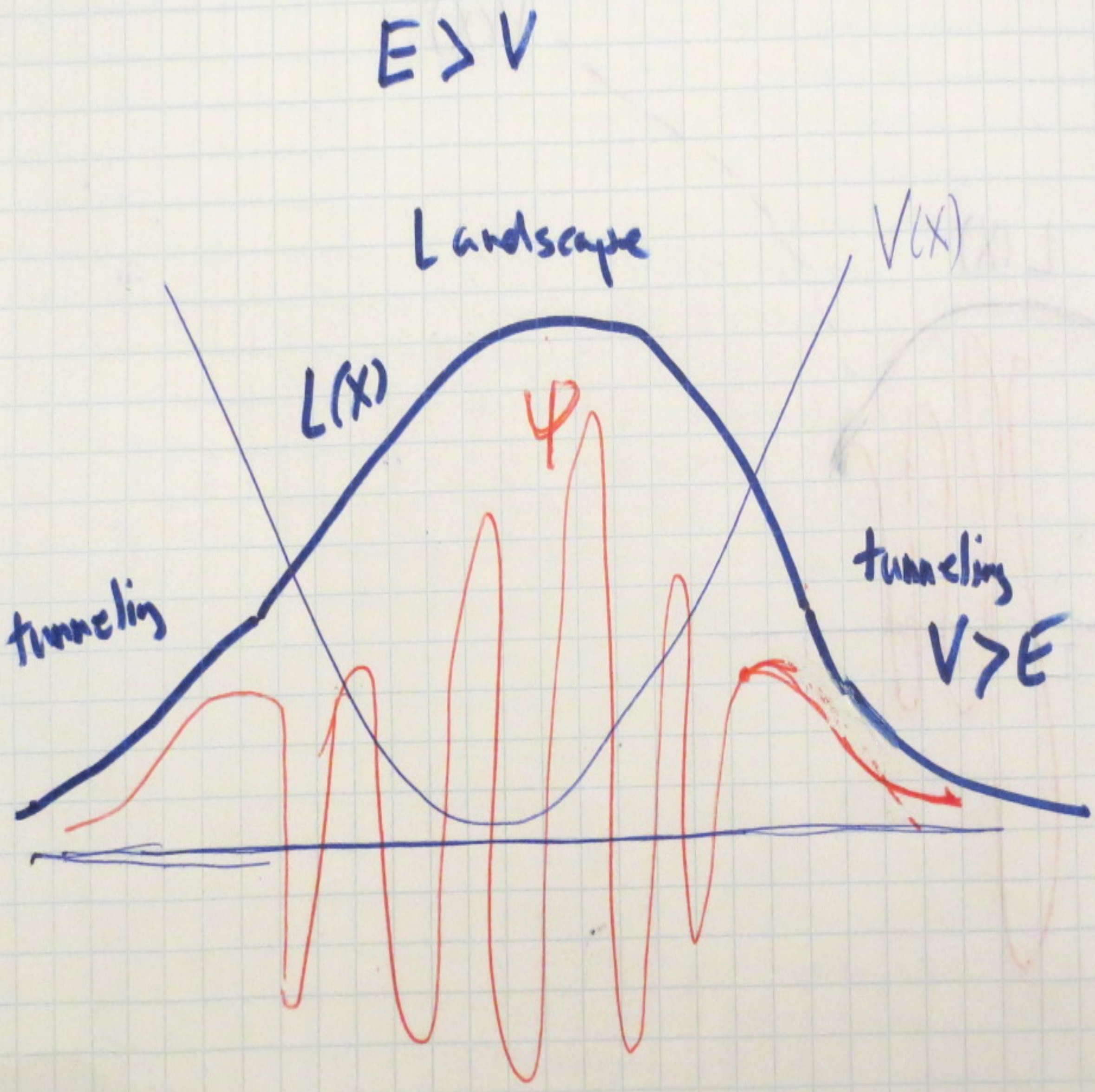
- Suppose that $V(x) \geq V_0 + b^2x^2$ on some interval I. Let

$$L(x) := \frac{2}{2b+V_0}\left(e^{-bx^2/2} + \frac{1}{2}\right)$$

and calculate to find H L ≥ 1. It follows from the maximum principle argument given above that:

$$|\psi(x)| - L(x) + \frac{1}{2b + V_0} = |\psi(x)| - \frac{2}{2b + V_0} e^{-bx^2/2}$$

Does not have an interior maximum on I.


E > V
Landscape
V(x)
L(x)
ψ
tunneling
tunneling
V>E