Singular Schrödinger operators with interactions supported by sets of codimension one

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The talk outline

- Setting the scene: why to consider singular Schrödinger operators
- δ-interactions supported by hypersurfaces
  - A simple definition
  - More general supports: Lipschitz partitions
  - Spectral properties: older and new results
- A more singular situation: δ'-interactions
  - Form definition
  - An operator inequality and its consequences
  - The strong δ' asymptotics
- General singular interactions
  - Definition
  - Operator inequalities again
  - Spectral properties
- Some open questions

Operators to deal with

The simplest example of the singular Schrödinger operators we are going to consider here can formally written as

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \quad \alpha > 0\,,$$

in $L^2(\mathbb{R}^n)$, where Γ is a zero-measure subset of $\mathbb{R}^n$, for instance, a manifold, a metric graph, etc.

Motivation: (a) Interesting mathematical objects, in particular, since their spectral properties reflect the geometry of Γ
(b) a useful model of *quantum graphs* and *generalized graphs* with the advantage that tunneling between edges is not neglected

We are going a wider class of operators in several respects

- the coupling strength may vary along the interaction support
- δ may be replaced by other, more singular interactions
- on the other hand, we restrict ourselves to the situations with $\operatorname{codim}\Gamma = 1$. Note that there are various results for $\operatorname{codim}\Gamma = 2$, cf. [E-Kondej'02,'15; E-Frank'07], while the remaining nontrivial case $\operatorname{codim}\Gamma = 3$ has not been studied so far

The δ-interaction

A natural tool to define the corresponding singular Schrödinger operator is to employ the appropriate quadratic form, namely

$$q_{\delta,\alpha}[\psi] := \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} - \alpha\|f|_\Gamma\|^2_{L^2(\Gamma)}$$

with the domain $H^1(\mathbb{R}^d)$ and to use the first representation theorem.

If Γ is a *smooth manifold* with $\operatorname{codim}\Gamma = 1$ one can easily check that the form defines a unique self-adjoint operator $H_{\alpha,\Gamma}$, which can alternatively characterized by boundary conditions: it acts as $-\Delta$ on functions from $H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, which are continuous and exhibit a normal-derivative jump,

$$\frac{\partial\psi}{\partial n}(x)\Big|_+ - \frac{\partial\psi}{\partial n}(x)\Big|_- = -\alpha(x)\psi(x)$$

This explains the formal expression as describing the attractive δ-interaction of strength $\alpha(x)$ perpendicular to Γ at the point x.

Alternatively, one sometimes uses the symbol $-\Delta_{\delta,\alpha}$ for this operator.

A wider class of interaction supports

The class of Γ mentioned above is rather narrow. To get a wider family we start from the following definition:

A finite family of Lipschitz domains $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ is called a *Lipschitz partition* of $\mathbb{R}^d$, $d \geq 2$, if

$$\mathbb{R}^d = \bigcup_{k=1}^n \overline{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_l = \emptyset\,, \quad k,l = 1,2,\dots,n, \;\; k \neq l\,.$$

The union $\cup_{k=1}^n \partial\Omega_k =: \Gamma$ is the *boundary* of $\mathcal{P}$. For $k \neq l$ we set $\Gamma_{kl} := \partial\Omega_k \cap \partial\Omega_l$ and we say that Ω_k and Ω_l, $k \neq l$, are neighboring domains if $\sigma_k(\Gamma_{kl}) > 0$, where σ_k is the Lebesgue measure on $\partial\Omega_k$.

Using standard coloring maps, we define the *chromatic number* $\chi_{\mathcal{P}}$ of $\mathcal{P}$ as the smallest number of colors allowed by the partition ‘map’. In particular, we know that $\chi_{\mathcal{P}} \leq 4$ if $d = 2$.

The δ-interaction

Then we have the following result [Behrndt-E-Lotoreichik'14]:

Proposition

Let $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ be a Lipschitz partition of $\mathbb{R}^d$ with the boundary Γ, and let $\alpha : \Gamma \to \mathbb{R}$ belong to $L^\infty(\Gamma)$. Then the quadratic form $q_{\delta,\alpha}$ defined above is closed and semibounded from below.

and consequently, there is a unique self-adjoint operator $-\Delta_{\delta,\alpha}$ associated with the form $q_{\delta,\alpha}$ which will be our object of interest.

Note that the interaction support may be a *proper subset* of Γ, since α may vanish on a part of Γ, hence it may be, e.g., a finite non-closed curve, a manifold with a boundary, etc.

Spectrum of $-\Delta_{\delta,\alpha}$

The spectrum is determined both by the *geometry of* Γ and the coupling function α, in particular, by its *sign*.

If Γ is *compact*, it is easy to see that $\sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = \mathbb{R}_+$.

On the other hand, the essential spectrum may change if the support Γ is non-compact. As an example, take a line in the plane and suppose that α is *constant and positive*; by separation of variables we find easily that $\sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = [-\frac{1}{4}\alpha^2, \infty)$.

The question about the *discrete spectrum* is more involved. Suppose first that interaction support is *finite*, $|\Gamma| < \infty$. It is clear that $\sigma_{\mathrm{disc}}(-\Delta_{\delta,\alpha})$ is empty if the interaction is repulsive, $\alpha \le 0$.

Spectrum of $-\Delta_{\delta,\alpha}$

On the other hand, the existence of a negative discrete spectrum for an attractive coupling is *dimension dependent*.

Consider for simplicity a constant α. For $d = 2$ bound states then exist whenever $|\Gamma| > 0$, in particular, we have a weak-coupling expansion, cf. [Kondej-Lotoreichik'14]

$$\lambda(\alpha) = \big(C_\Gamma + o(1)\big)\, \exp\left(-\frac{4\pi}{\alpha|\Gamma|}\right) \quad \text{as} \quad \alpha|\Gamma| \to 0+$$

On the other hand, for $d = 3$ the singular coupling must exceed a critical value. As an example, let Γ be a sphere of radius $R > 0$ in $\mathbb{R}^3$, then by [Antoine-Gesztesy-Shabani'87] we have

$$\sigma_{\mathrm{disc}}(H_{\alpha,\Gamma}) \neq \emptyset \quad \text{iff} \quad \alpha R > 1$$

and the same obviously holds in dimensions $d > 3$.

A δ-interaction supported by infinite curves

A geometrically induced discrete spectrum may exist even if Γ is infinite and $\inf \sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) < 0$. Consider, for instance, a *non-straight, piecewise C^1-smooth curve* $\Gamma : \mathbb{R} \to \mathbb{R}^2$ parameterized by its arc length, $|\Gamma(s) - \Gamma(s')| \le |s - s'|$, assuming in addition that

- $|\Gamma(s) - \Gamma(s')| \ge c|s - s'|$ holds for some $c \in (0, 1)$
- Γ is *asymptotically straight*: there are $d > 0$, $\mu > \frac{1}{2}$ and $\omega \in (0, 1)$ such that

$$1 - \frac{|\Gamma(s) - \Gamma(s')|}{|s - s'|} \le d \left[1 + |s + s'|^{2\mu}\right]^{-1/2}$$

in the sector $S_\omega := \left\{(s, s') : \omega < \frac{s}{s'} < \omega^{-1}\right\}$

Theorem (E-Ichinose'01)

Under these assumptions, $\sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = [-\frac{1}{4}\alpha^2, \infty)$ and $-\Delta_{\delta,\alpha}$ has at least one eigenvalue below the threshold $-\frac{1}{4}\alpha^2$.

Geometrically induced bound states, continued

- The result is obtained via (generalized) Birman-Schwinger principle regarding the bending a *perturbation of the straight line*
- the crucial observation is that – in view of the 2D free resolvent kernel properties – this perturbation is *sign definite* and *compact*
- *Higher dimensions*: the situation is more complicated. For *smooth curved surfaces* $\Gamma \subset \mathbb{R}^3$ an analogous result is proved in the strong coupling asymptotic regime, $\alpha \to \infty$, only
- On the other hand, we have an example of a *conical surface* of an opening angle $\theta \in (0, \frac{1}{2}\pi)$ in $\mathbb{R}^3$, where for any constant $\alpha > 0$ we have $\sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = \mathbb{R}_+$ and an *infinite numbers of negative eigenvalues* accumulating at zero, cf. [Behrndt-E-Lotoreichik'14]
- Moreover, the above result remain valid for any *local deformation* of the conical surface. We also know the accumulation rate for conical layers: by [Lotoreichik–Ourmières-Bonafos'16] it is

$$\mathcal{N}_{-\frac{1}{4}\alpha^2 - E}(-\Delta_{\delta,\alpha}) \sim \frac{\cot\theta}{4\pi}\,|\ln E|\,, \quad E \to 0+\,.$$

Geometrically induced bound states, continued

- On the other hand, the result is again dimension-dependent: for a conical surface in $\mathbb{R}^d$, $d > 3$, we have $\sigma_{\rm disc}(-\Delta_{\delta,\alpha}) = \emptyset$, cf. [Lotoreichik–Ourmières-Bonafos'16].
- Implications for *more complicated Lipschitz partitions*: let $\tilde\Gamma \supset \Gamma$ holds in the set sense, then $H_{\alpha,\tilde\Gamma} \le H_{\alpha,\Gamma}$. If the essential spectrum thresholds are the same – which is often easy to establish – then $\sigma_{\rm disc}(H_{\alpha,\tilde\Gamma}) \ne \emptyset$ whenever the same is true for $\sigma_{\rm disc}(H_{\alpha,\Gamma})$
- Many other results, for instance, concerning the *strong coupling asymptotics*: for a C^4 smooth *curve in $\mathbb{R}^2$ without ends* the j-th eigenvalue of $-\Delta_{\delta,\alpha}$ behaves as

$$\lambda_j(\alpha) = -\frac{\alpha^2}{4} + \mu_j + \mathcal{O}(\alpha^{-1}\ln\alpha)$$

in the limit $\alpha \to \infty$, where μ_j is the j-th ev of $S_\Gamma = -\frac{\rm d}{{\rm d}s^2} - \frac{1}{4}\kappa(s)^2$ on $L^2((0,|\Gamma|))$, where κ is the *signed curvature* of Γ.

Geometrically induced bound states, continued

- The same is true for curves with *regular* ends; the comparison operator S_Γ is then subject to *Dirichlet* boundary conditions, cf. [E-Pankrashkin'14].
- Similar results are valid C^4 smooth *surfaces in* $\mathbb{R}^3$; here the comparison operator is $S_\Gamma = -\Delta_\Gamma + K - M^2$, where $-\Delta_\Gamma$ is Laplace-Beltrami operator on Γ and K, M, respectively, are the corresponding *Gauss* and *mean* curvatures. For surfaces with a boundary additional technical assumptions are needed, cf. [Dittrich-E-Kühn-Pankrashkin'16].
- For infinite curves in $\mathbb{R}^2$ we have also a *weak bending asymptotics*: for a family Γ_θ parametrized by the bending angle θ one proves $\lambda(H_{\alpha,\Gamma_\theta}) = -\frac{1}{4}\alpha^2 + a\theta^4 + o(\theta^4)$ with an explicit $a < 0$ as $\theta \to 0+$ under some technical assumptions [E-Kondej'16]. In particular, for broken line we have $a = -\frac{\alpha^2}{36\pi^2}$.
- Also various other results are known ...

More singular operators: the δ'-interaction

Having in mind the one-dimensional point interaction, we can define for a smooth planar curve the operator $-\Delta_{\delta',\beta}$ using boundary conditions: it acts as Laplacian outside the interaction support,

$$(H_{\beta,\Gamma}\psi)(x) = -(\Delta\psi)(x)\,, x \in \mathbb{R}^2 \setminus \Gamma\,,$$

with the domain consisting of functions $\psi \in H^2(\mathbb{R}^2 \setminus \Gamma)$ that satisfy the b.c. $\partial_{n_\Gamma}\psi(x) = \partial_{-n_\Gamma}\psi(x) =: \psi'(x)|_\Gamma,\ -\beta\psi'(x)|_\Gamma = \psi(x)|_{\partial_+\Gamma} - \psi(x)|_{\partial_-\Gamma}\}$, where n_Γ is the normal to Γ and $\psi(x)|_{\partial_\pm\Gamma}$ are the appropriate traces.

The corresponding quadratic form is easily seen to be

$$h_{\beta,\Gamma}[\psi] = \|\nabla\psi\|^2 - \beta^{-1}\int_\Gamma |\psi(s,0_+) - \psi(s,0_-)|^2\,\mathrm{d}s$$

defined on functions $\psi \in H^1(\mathbb{R}^2 \setminus \Gamma)$ as $\psi(s,u)$, where s, u are the natural curvilinear coordinates in the vicinity of Γ. This can be used to define the δ'-interaction in other dimensions and for more general Lipschitz partitions.

Note that the strong-coupling in this case means $\beta \to 0+$.

The δ'-interaction

Let $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ be a Lipschitz partition of $\mathbb{R}^d$ with the boundary Γ, and let $\beta : \Gamma \to \mathbb{R}$ be such that $\beta^{-1} \in L^\infty(\Gamma)$. Then we define the form

$$q_{\delta',\beta}[f,g] := \sum_{k=1}^{n} \big(\nabla f_k, \nabla g_k\big)_{L^2(\Omega_k}$$
$$- \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \big(\beta_{kl}^{-1}(f_k|_{\Gamma_{kl}} - f_l|_{\Gamma_{kl}}), g_k|_{\Gamma_{kl}} - g_l|_{\Gamma_{kl}}\big)_{L^2(\Gamma_{kl})}$$

with the domain $\bigoplus_{k=1}^n H^1(\Omega_k)$; we denote here $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ for $k, l = 1, 2, \dots, n$, $k \neq l$, and β_{kl} means the restrictions of β to Γ_{kl}.

As in the δ case, we have the following result [Behrndt-E-Lotoreichik'14]:

Proposition

The form $q_{\delta',\beta}$ is closed and semibounded from below.

The s-a operator associated with $q_{\delta',\beta}$ will be denoted as $-\Delta_{\delta',\beta}$ or $H_{\beta.\Gamma}$

Spectrum of $-\Delta_{\delta',\beta}$

Similarly to the δ case, we have $\sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta}) = \mathbb{R}_+$ if Γ is compact.

A δ'-interaction supported by a non-compact Γ, on the other hand, may change the essential spectrum; an example is again a line in the plane with a constant and positive β, where by separation of variables we find $\sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta}) = [-\frac{4}{\beta^2}, \infty)$.

It is also clear that the a compactly supported δ'-interaction can give rise to a nontrivial discrete spectrum only if *it is not (purely) repulsive*.

On the other hand, relations between the discrete spectrum and the form of Γ are, in general, different from the δ situation. It is now the *topology* of the interaction support which plays role.

The δ' interaction in the plane

Consider a finite curve Γ in $\mathbb{R}^2$. If it is a *loop*, then it is easy to see that $\sigma_{\mathrm{disc}}(-\Delta_{\delta',\beta}) \neq \emptyset$ for any constant $\beta > 0$: just try a trial function which is a constant inside the loop and zero otherwise.

On the other hand, by [M. Dauge, private communication] we have

Proposition

If Γ is not closed, there is a $\beta_0 > 0$ such that $\sigma_{\mathrm{disc}}(-\Delta_{\delta',\beta}) = \emptyset$ holds for all constant $\beta > \beta_0$.

For a class of Γ we have a quantitative result, namely for those that are nonclosed, piecewise C^1, and *monotone*, i.e. allow a parametrisation by a piecewise C^1 map $\varphi : (0, R) \to \mathbb{R}$,

$$\Gamma = \left\{x_0 + r(\cos\varphi(r), \sin\varphi(r)) :\; r \in (0, R)\right\}$$

Theorem (Jex-Lotoreichik'16)

We have $\sigma(-\Delta_{\delta',\beta}) \subset \mathbb{R}_+$ if $\beta > 2\pi r\sqrt{1 + (r\varphi'(r))^2}$ for all $r \in (0, R)$.

An operator inequality

Spectral analysis of $-\Delta_{\delta',\beta}$ is more difficult because we lack a direct counterpart to some of the tools used before, in particular, to the (generalized) Birman-Schwinger principle.

One the other hand, there is a useful relation between the two cases:

Theorem (Behrndt-E-Lotoreichik'14)

Let $\mathcal{P} = \{\Omega_k\}_{k=1}^n$ be a Lipschitz partition of $\mathbb{R}^d$ with boundary Γ and chromatic number $\chi_{\mathcal{P}}$. Let $\alpha, \beta \colon \Gamma \to \mathbb{R}$ be such that $\alpha, \beta^{-1} \in L^\infty(\Gamma)$ and assume that

$$0 < \beta \leq \frac{4}{\alpha} \sin^2\left(\frac{\pi}{\chi_{\mathcal{P}}}\right).$$

Then there exists a unitary operator $U \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that the self-adjoint operators $-\Delta_{\delta,\alpha}$ and $-\Delta_{\delta',\beta}$ satisfy the inequality

$$U^{-1}(-\Delta_{\delta',\beta})U \leq -\Delta_{\delta,\alpha}.$$

Sketch of the proof

By assumption, to the given $\mathcal{P}$ there is an optimal *coloring map*

$$\phi\colon \{1,2,\dots,n\} \to \{0,1,\dots,\chi_{\mathcal{P}}-1\}$$

such that for any $k \neq l$ such that $\sigma_k(\Gamma_{kl}) > 0$ we have $\phi(k) \neq \phi(l)$.

Then we define n complex numbers $\mathcal{Z} := \{z_k\}_{k=1}^n$ on the unit circle,

$$z_k := \exp\left(i\frac{2\pi\phi(k)}{\chi_{\mathcal{P}}}\right), \quad k = 1,2,\dots,n\,;$$

it is easy to see that for $k \neq l$ such that $\sigma_k(\Gamma_{kl}) > 0$ they satisfy

$$|z_k - z_l|^2 \geq 2 - 2\cos\left(\tfrac{2\pi}{\chi_{\mathcal{P}}}\right),$$

in other words $4\sin^2\left(\frac{2\pi}{\chi_{\mathcal{P}}}\right) \leq |z_k - z_l|^2$.

Sketch of the proof

Putting now $\alpha_{\mathcal{Z}}(x) := |z_k - z_l|^2 \beta_{kl}^{-1}(x)$ for $x \in \Gamma_{kl}$ with $k \neq l$, we find

$$0 < \alpha \leq \frac{4}{\beta} \sin^2\left(\tfrac{2\pi}{\chi_{\mathcal{P}}}\right) \leq \alpha_{\mathcal{Z}}\,.$$

Now we define the unitary operator $U_{\mathcal{Z}} \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(U_{\mathcal{Z}} f)(x) := z_k f_k(x)\,, \quad x \in \Omega_k\,, \quad k = 1, \dots, n\,.$$

Using then the above inequality in combination with the explicit expressions of the involved quadratic forms, it is not difficult to derive the sought result.

Consequences of the inequality

The above result allows to draw conclusions from an operator comparison.

Denote by $\{\lambda_k(-\Delta_{\delta,\alpha})\}_{k=1}^{\infty}$ and $\{\lambda_k(-\Delta_{\delta',\beta})\}_{k=1}^{\infty}$ the eigenvalues of the operators $-\Delta_{\delta,\alpha}$ and $-\Delta_{\delta',\beta}$, respectively, below the bottom of their essential spectra, enumerated in non-decreasing order and repeated with multiplicities, and let $N(-\Delta_{\delta,\alpha})$ and $N(-\Delta_{\delta',\beta})$ be their total numbers.

Corollary

Under the assumption of the theorem, we have

(i) $\lambda_k(-\Delta_{\delta',\beta}) \le \lambda_k(-\Delta_{\delta,\alpha})$ *for all* $k \in \mathbb{N}$;

(ii) $\min \sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta}) \le \min \sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha})$;

(iii) *If* $\min \sigma_{\mathrm{ess}}(-\Delta_{\delta,\alpha}) = \min \sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta})$, *then* $N(-\Delta_{\delta,\alpha}) \le N(-\Delta_{\delta',\beta})$.

Consequences of the inequality

The estimates are the better the smaller the chromatic number is.

Corollary

Under the stated assumptions, let $\chi_{\mathcal{P}} = 2$ *and* $0 < \beta \le \frac{4}{\alpha}$, *then there is a unitary operator such that*

$$U^{-1}(-\Delta_{\delta',\beta})U \le -\Delta_{\delta,\alpha}\,,$$

and consequently, the conclusions of the previous corollary are valid.

Moreover, the examples with Γ being a line in the plane show that the inequality $0 < \beta \le \frac{4}{\alpha}$ cannot be improved.

Example: Let Γ be a bent, asymptotically straight curve considered above, now supporting the δ'-interaction with a constant $\beta > 0$. Choose $\alpha = \frac{4}{\beta}$, then $-\Delta_{\delta',\beta}$ and $-\Delta_{\delta,\alpha}$ have the same essential spectrum. Since we know that $\sigma_{\mathrm{disc}}(-\Delta_{\delta,\alpha}) \ne \emptyset$, *the same is true for* $-\Delta_{\delta',\beta}$.

Strong coupling on a δ' loop

Some δ arguments, though, can be adapted easily to the δ' situation.

Theorem (E-Jex'13)

Let Γ be a C^4-smooth closed curve without self-intersections. Then $\sigma_{\mathrm{ess}}(H_{\beta,\Gamma}) = [0,\infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that $\#\sigma_{\mathrm{disc}}(H_{\beta,\Gamma}) \geq n$ holds for $\beta \in (0,\beta_n)$. Denoting by $\lambda_j(\beta)$ the j-th eigenvalue of $H_{\beta,\Gamma}$, counted with multiplicity, we have the expansion

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}\big(\beta|\ln\beta|\big)\,, \quad j = 1,\dots,n\,,$$

valid as $\beta \to 0_+$, where μ_j is the j-th eigenvalue of the comparison operator S_Γ, the same as before. Moreover, for the counting function $\beta \mapsto \#\sigma_d(H_{\beta,\Gamma})$ we have

$$\sharp\sigma_{\mathrm{disc}}(H_{\beta,\Gamma}) = \frac{2L}{\pi\beta} + \mathcal{O}(|\ln\beta|) \quad \text{as } \beta \to 0_+\,.$$

A similar result holds for infinite curves, cf. [Jex'14], and for strong δ' interaction supported by surfaces *without boundary*, cf. [E-Jex'14]

More general interactions

The δ and δ' are just particular cases of the general, *four-parameter family* of point interactions, and we are now going to construct singular Schrödinger operators with such a general interaction.

For simplicity we restrict ourselves to the simplest partition of the space, namely we assume that $\Gamma \subset \mathbb{R}^d$, $d \geq 2$, is the boundary of a (bounded or unbounded) Lipschitz domain $\Omega = \Omega_\mathrm{i}$ and $\Omega_\mathrm{e} := \mathbb{R}^d \setminus (\Omega_\mathrm{i} \cup \Gamma)$; for $f \in L^2(\mathbb{R}^d)$ we write $f_j = f|_{\Omega_j}$, $j = \mathrm{i}, \mathrm{e}$, and $f = f_\mathrm{i} \oplus f_\mathrm{e}$.

The trace of $f \in H^1(\Omega_j)$ on Γ is denoted by $f|_\Gamma \in H^{1/2}(\Gamma)$. For each $f \in H^1(\Omega_j)$ we define the derivative of f with respect to the outer unit normal on $\Gamma = \partial\Omega_j$ using Green's first identity; if Γ is sufficiently smooth and f is differentiable up to the boundary then $\partial_{\nu_j} f|_\Gamma$ is the usual derivative. The outer unit normals for Ω_i and Ω_e coincide up to a minus sign, in particular, for $f \in H^2(\mathbb{R}^d)$ we have $\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma = 0$.

More general interactions

The conditions defining the general point interaction can be written in different form. We employ the one from [E-Grosse'99], up to signs, which has the advantage of making the particular cases of δ and δ' visible.

The interactions supported on Γ will be thus described by Laplacian on $\mathbb{R}^d \setminus \Gamma$ subject to the interface conditions

$$\begin{aligned} \partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma &= \frac{\alpha}{2}\left(f_\mathrm{i}|_\Gamma + f_\mathrm{e}|_\Gamma\right) + \frac{\gamma}{2}\left(\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma - \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma\right), \\ f_\mathrm{i}|_\Gamma - f_\mathrm{e}|_\Gamma &= -\frac{\overline{\gamma}}{2}\left(f_\mathrm{i}|_\Gamma + f_\mathrm{e}|_\Gamma\right) + \frac{\beta}{2}\left(\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Gamma - \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Gamma\right). \end{aligned}$$

Concerning the *coefficient functions*, we assume that $\alpha : \Gamma \to \mathbb{R}$ and $\gamma : \Gamma \to \mathbb{C}$ are bounded, measurable functions. Moreover, let $\Gamma_\beta \subset \Gamma$ be a relatively open subset and let $\beta : \Gamma \to \mathbb{R}$ be a function such that β^{-1} is measurable and bounded on Γ_β and $\beta = 0$ identically on $\Gamma_0 := \Gamma \setminus \Gamma_\beta$.

For some of them, however, the above conditions are formal and we have to seek an alternative way to define the operators in question.

The quadratic form definition

We employ again a suitable quadratic form. Given $\mathcal{A} = \begin{pmatrix} \alpha & \gamma \\ -\bar{\gamma} & \beta \end{pmatrix}$ we define the symmetric matrix function $\Theta_{\mathcal{A}}$ on Γ by

$$\Theta_{\mathcal{A}} = \begin{pmatrix} \frac{|1+\frac{\gamma}{2}|^2}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} & \frac{(\frac{\bar{\gamma}}{2}-1)(1+\frac{\gamma}{2})}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} \\ \frac{(\frac{\gamma}{2}-1)(1+\frac{\bar{\gamma}}{2})}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} & \frac{|1-\frac{\gamma}{2}|^2}{\beta}\mathbb{I}_{\Gamma_\beta} + \frac{\alpha}{4} \end{pmatrix}$$

with the convention that $\frac{1}{\beta}\mathbb{I}_{\Gamma_\beta}$ equals zero on Γ_0.

Then we define a quadratic form $h_{\mathcal{A}}$ in $L^2(\mathbb{R}^d)$ in the following way,

$$q_{\mathcal{A}}(f,g) = \int_{\Omega_\mathrm{i}} \nabla f_\mathrm{i} \cdot \overline{\nabla g_\mathrm{i}}\,\mathrm{d}x + \int_{\Omega_\mathrm{e}} \nabla f_\mathrm{e} \cdot \overline{\nabla g_\mathrm{e}}\,\mathrm{d}x - \int_\Gamma \left\langle \Theta_{\mathcal{A}} \begin{pmatrix} f_\mathrm{i} \\ f_\mathrm{e} \end{pmatrix}, \begin{pmatrix} g_\mathrm{i} \\ g_\mathrm{e} \end{pmatrix} \right\rangle \mathrm{d}\sigma\,,$$

$$\mathcal{D}(q_{\mathcal{A}}) = \left\{ f_\mathrm{i} \oplus f_\mathrm{e} \in H^1(\Omega_\mathrm{i}) \oplus H^1(\Omega_\mathrm{e}) : (1+\tfrac{\bar{\gamma}}{2})f_\mathrm{i} = (1-\tfrac{\bar{\gamma}}{2})f_\mathrm{e} \text{ on } \Gamma_0 \right\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{C}^2$ and σ is the surface measure on Γ. Note that $q_{\mathcal{A}}$ is well-defined since the entries of $\Theta_{\mathcal{A}}$ are bounded functions.

The quadratic form definition

Under the stated assumption we have [E-Rohleder'16]:

Proposition

The form $q_{\mathcal{A}}$ in $L^2(\mathbb{R}^d)$ is densely defined, symmetric, semibounded below and closed.

Hence there is a unique selfadjoint, semibounded operator $-\Delta_{\mathcal{A}}$ associated with $q_{\mathcal{A}}$; it the coefficients are regular enough it coincides with the Laplacian subject to the above stated interface conditions.

Remark: The definition includes not only the δ- ($\beta = \gamma = 0$) and δ'-interaction ($\alpha = \gamma = 0$), but also other cases of interest. For instance, given real constants $c_{\rm i}$, $c_{\rm e}$ with $c_{\rm i} + c_{\rm e} \neq 0$ and choosing

$$\alpha = \frac{4c_{\rm i}c_{\rm e}}{c_{\rm i}+c_{\rm e}}, \quad \beta = \frac{4}{c_{\rm i}+c_{\rm e}}, \quad \gamma = \frac{2(c_{\rm i}-c_{\rm e})}{c_{\rm i}+c_{\rm e}},$$

we get separated regions with Robin conditions, $\partial_{\nu_j} f_j = c_j f_j\,,\ j = {\rm i}, {\rm e}$.

Spectral properties of $-\Delta_{\mathcal{A}}$

A lot can be said about spectrum of $-\Delta_{\mathcal{A}}$, let us mention a few results.

Theorem (E-Rohleder'16)

Let $\Omega_{\rm i}$ be bounded, i.e. Γ is compact. Then the resolvent difference

$$(-\Delta_{\mathcal{A}}-\lambda)^{-1}-(-\Delta_{\rm free}-\lambda)^{-1}, \quad \lambda\in\rho(-\Delta_{\mathcal{A}})\cap\rho(-\Delta_{\rm free})$$

is compact. In particular, $\sigma_{\rm ess}(-\Delta_{\mathcal{A}})=\mathbb{R}_+$ and the discrete spectrum $\sigma(-\Delta_{\mathcal{A}})\cap(-\infty,0)$ is finite.

Concerning the existence of $\sigma_{\rm disc}(-\Delta_{\mathcal{A}})$, in the presence of δ' we have the following sufficient condition:

Theorem (E-Rohleder'16)

In addition the hypotheses of the previous theorem, let $\Gamma=\Gamma_\beta$, i.e., $\beta(s)\neq 0$ for all $s\in\Gamma$. If $\int_\Gamma\left(\frac{|1+\frac{\gamma}{2}|^2}{\beta}+\frac{\alpha}{4}\right)\mathrm{d}\sigma>0$ holds, $N(-\Delta_{\mathcal{A}})>0$.

Spectral properties of $-\Delta_{\mathcal{A}}$

In the absence of δ' the claim depends on the dimension:

Theorem (E-Rohleder'16)

Let Γ be compact in dimension $d = 2$. Assume that $\beta = 0$ identically on Γ, and moreover, $\alpha(s) \geq \alpha_{\text{min}} > 0$ for all $s \in \Sigma$ and let $\gamma \in \mathbb{C}$ be constant. Then $N(-\Delta_{\mathcal{A}}) > 0$.

If $d \geq 3$ the situation is different:

Proposition

Let Γ be compact, $d \geq 3$, and $\beta = 0$ identically on Γ. Moreover, let $0 \leq \alpha(s) \leq \alpha_{\text{max}}$ for all $s \in \Sigma$ and let $\gamma \in \mathbb{C}$ be constant. Define

$$\widetilde{\alpha} = \frac{\alpha_{\text{max}}}{\min\{|1+\gamma/2|^2, |1-\gamma/2|^2\}} \geq 0$$

and let $-\Delta_{\delta,\widetilde{\alpha}}$ be the Schrödinger operator in $L^2(\mathbb{R}^d)$ with δ-interaction of strength $\widetilde{\alpha}$ on Γ. If $N(-\Delta_{\delta,\widetilde{\alpha}}) = 0$ the same is true for $N(-\Delta_{\mathcal{A}})$.

Spectral properties of $-\Delta_{\mathcal{A}}$

The situation is more complicated if Γ is non-compact:

Theorem (E-Rohleder'16)

Let Γ be a surface in $\mathbb{R}^3$ homeomorphic to the plane which is C^2 smooth outside a compact and asymptotically planar in the sense that K, M vanish asymptotically. Suppose further that the functions α, β, γ are constant outside a compact and $\alpha(s), \beta(s)$ are nonnegative for all $s \in \Gamma$, then under additional mild assumptions we have $\sigma_{\rm ess}(-\Delta_{\mathcal{A}}) \subset [m_{\mathcal{A}}, \infty)$, where

$$m_{\mathcal{A}} = \begin{cases} -\frac{4\alpha^2}{(4+|\gamma|^2)^2}, & \text{if } \beta = 0 \\ -\frac{\left(4+\det\mathcal{A}+\sqrt{-16\alpha\beta+(4+\det\mathcal{A})^2}\right)^2}{16\beta^2} & \text{if } \beta \neq 0. \end{cases}$$

and α, β, γ are the constant function values outside the compact.

In some case one can prove equality, $\sigma_{\rm ess}(-\Delta_{\mathcal{A}}) = [m_{\mathcal{A}}, \infty)$, for instance if Γ is a plane outside a compact.

Operator inequalities

To prove the existence of a non-void discrete spectrum one can combine known results in particular case with *operator inequalities*. In various particular situations one can prove the existence of a unitary operator, denoted generically as U, which make it possible:

Suppose again that $\alpha : \Gamma \to \mathbb{R}$ and $\gamma : \Gamma \to \mathbb{C}$ are bounded, measurable functions, and $\beta : \Gamma \to \mathbb{R}$ is measurable with β^{-1} bounded, then:

(a) Let $\mathcal{A} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with $\alpha(s) \geq 0$ and $\beta(s) > 0$ for all $s \in \Gamma$. Let further $\alpha(s) \leq \frac{4}{\beta(s)}$ for all $s \in \Gamma$, then

$$U^*(-\Delta_{\mathcal{A}})U \leq -\Delta_{\delta,\alpha}\,.$$

(b) Let $\mathcal{A} = \begin{pmatrix} 0 & \gamma \\ -\bar{\gamma} & \beta \end{pmatrix}$ with $\beta(s) > 0$ for all $s \in \Gamma$ and $\gamma \in i\mathbb{R}$ being a constant. Let further $\widetilde{\alpha} : \Gamma \to \mathbb{R}$ be measurable and bounded satisfying $\widetilde{\alpha}(s) \leq \frac{4+|\gamma|^2}{\beta(s)}$ for all $s \in \Gamma$, then

$$U^*(-\Delta_{\mathcal{A}})U \leq -\Delta_{\delta,\alpha}\,.$$

Operator inequalities

(c) Let $\mathcal{A} = \begin{pmatrix} \alpha & \gamma \\ -\bar{\gamma} & 0 \end{pmatrix}$ with $\alpha(s) \geq 0$ for all $s \in \Gamma$ and $\gamma \in i\mathbb{R}$ being a constant. Let further $\widetilde{\alpha} : \Gamma \to \mathbb{R}$ be measurable and bounded satisfying $\widetilde{\alpha}(s) \leq \frac{\alpha(s)}{|1+\frac{\gamma}{2}|^2}$ for all $s \in \Gamma$, then

$$U^*(-\Delta_{\mathcal{A}})U \leq -\Delta_{\delta,\alpha}\,.$$

(d) Let $\mathcal{A} = \begin{pmatrix} \alpha & \gamma \\ -\bar{\gamma} & 0 \end{pmatrix}$ with $\alpha(s) \geq 0$ for all $s \in \Gamma$ and $\gamma : \Gamma \to \mathbb{C}$ being measurable and bounded. Let further $\widetilde{\beta} : \Gamma \to \mathbb{R}$ be such that $\widetilde{\beta}^{-1}$ is measurable and bounded satisfying $\alpha(s) \leq \frac{4}{\widetilde{\beta}(s)}$ for all $s \in \Gamma$, then

$$U^*(-\Delta_{\delta',\widetilde{\beta}})U \leq -\Delta_{\mathcal{A}}\,.$$

The first three can be used to estimate the spectra from the known results about the δ-interaction, the last one includes also the *intermediate class* which occurs if $\mathrm{Re}\,\gamma \neq 0$.

Open questions

In my view, the main challenge concerns the strong-coupling behavior in situations with *less regularity*, in the first place such a behavior for Hamiltonians of *branched leaky graphs.*

Conjecture: The strong coupling limit of broken curves/branched graphs behaves *similarly to shrinking Dirichlet networks or tubes*, i.e. a nontrivial limit with the natural energy renormalization can be obtained provided the system exhibits a *threshold resonance*.

For *periodic manifolds* the absolute continuity of the spectrum is not proven even in the δ-interaction case, except a non-uniform, strong coupling result – to say nothing of the more singular interactions

Other problems: strong-coupling *asymptotic behavior of gaps* for periodic manifolds, a better understanding of the influence of regular potentials and *magnetic fields:* how do they influence curvature-induced bound states? We *conjecture* they may destroy them. Furthermore, where does the *mobility edge* lies if Γ is randomized?, etc., etc.

The talk sources

[Ex08] For results prior to 2008 I refer to P.E: Leaky quantum graphs: a review, *Proceedings of the Isaac Newton Institute programme "Analysis on Graphs and Applications"*, AMS "Proceedings of Symposia in Pure Mathematics" Series, vol. 77, Providence, R.I., 2008; pp. 523–564.

[BEL14] J. Behrndt, P.E., V. Lotoreichik: Schrödinger operators with δ and δ'-interactions on Lipschitz surfaces and chromatic numbers of associated partitions, *Rev. Math. Phys. A: Math. Theor.* **26** (2014), 1450015 (43pp)

[EJ13] P.E., M. Jex: Spectral asymptotics of a strong δ' interaction on a planar loop, *J. Phys. A: Math. Theor* **46** (2013), 345201

[ER16] P.E., J. Rohleder: Generalized interactions supported on hypersurfaces, *J. Math. Phys.* **57** (2016), 041507 (24pp)

[EV16] P.E., S. Vugalter: On the existence of bound states in asymmetric leaky wires, *J. Math. Phys.* **57** (2016), 022104 (15pp)

as well as the other papers mentioned in the course of the presentation.

It remains to say

Thank you for your attention!